Two-dimensional limits: methods and examples

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1 Summary

In this note, we give some examples of limits of functions defined on $\mathbb{R}^2$. We can use the following methods to show that a limit exists:

- Continuity
- The squeeze theorem
- Polar coordinates + the squeeze theorem

We can use the following methods to show that a limit does *not* exist:

- Checking along different paths: e.g., considering paths $y = mx$ and obtaining a limit that depends on m
- Polar coordinates: e.g. obtaining a limit that depends on θ when approaching along paths with fixed θ.

We emphasize that **checking along coordinate axes, lines, or polynomials is not sufficient to conclude that a limit exists**.

2 Using continuity

Recall that a function $f(x, y)$ is continuous at a point (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

Example 1. *Evaluate the following limit if it exists, or show that it does not exist:*

$$\lim_{(x,y)\to(1,1)} (x^2y + x\cos(x + y))$$

Solution: Recall that polynomials in x, y are continuous on $\mathbb{R}^2$, and $x \mapsto \cos x$ is continuous on $\mathbb{R}$. Since sums, products, and compositions of continuous functions are continuous, this f is continuous. We may therefore plug in $(1, 1)$, which gives

$$\lim_{(x,y)\to(1,1)} (x^2y + x\cos(x + y)) = 1 + \cos 2.$$

3 Using the squeeze theorem

The **squeeze theorem** is the following statement: if $f(x,y), g(x,y), h(x,y)$ are defined near (a,b) and satisfy

1. $g(x,y) \le f(x,y) \le h(x,y)$
2. $\lim_{(x,y)\to(a,b)} g(x,y) = A = \lim_{(x,y)\to(a,b)} h(x,y)$,

then the limit of f exists and $\lim_{(x,y)\to(a,b)} f(x,y) = A$.

Example 2. *Evaluate the following limit if it exists, or show that it does not exist:*

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\sin\left(\frac{1}{x^2+y^2}\right)$$

Solution: Using the fact that $|\sin x| \le 1$, we have

$$\left|\sqrt{x^2+y^2}\sin\left(\frac{1}{x^2+y^2}\right)\right| \le \sqrt{x^2+y^2}.$$

We can see that the limit of the function on the right-hand side is 0 by continuity. By the squeeze theorem (using $h(x,y) = \sqrt{x^2+y^2}$ and $g(x,y) = -\sqrt{x^2+y^2}$), the original limit is also 0.

4 Showing that limits do not exist

Example 3. *Evaluate the following limit if it exists, or show that it does not exist:*

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

Solution 1: Let f be the function in the limit above, defined for $(x,y) \ne (0,0)$. Consider lines through the origin of the form $y = mx$. For $x \ne 0$, we have

$$f(x,mx) = \frac{x^2}{x^2+m^2x^2} = \frac{1}{1+m^2}.$$

Since this varies with m, we obtain different limits as $(x,y) \to (0,0)$ along the lines $y = mx$. Therefore the limit does not exist.

Solution 2: We can also phrase the above argument in terms of polar coordinates. Write

$$f(r\cos\theta, r\sin\theta) = \frac{r^2\cos^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \cos^2\theta.$$

Since this depends on θ, we obtain different limits as $(x,y) \to (0,0)$ along lines of angle θ off the x-axis.

Example 4. *Evaluate the following limit if it exists, or show that it does not exist:*

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

Solution: Let f be the function in the limit above, defined for $(x,y) \neq (0,0)$. For $x \neq 0$, we have

$$f(x,mx) = \frac{x^2mx}{x^4+m^2x^2} = \frac{mx}{x^2+m^2} \to 0 \text{ as } x \to 0.$$

But we also have

$$f(x,mx^2) = \frac{x^2mx^2}{x^4+m^2x^4} = \frac{m}{1+m^2},$$

so the limit doesn't exist. This example illustrates that it is not sufficient, in general, to check only linear paths.

Example 5. *Evaluate the following limit if it exists, or show that it does not exist:*

$$\lim_{(x,y)\to(0,0)} \frac{x^ny}{x^{2n}+y^2},$$

where $n \geq 2$ is an integer.

Solution: This is a variation of the previous example. Let f be the function in the limit. If k is an integer such that $0 < k < n$, then for $x \neq 0$,

$$f(x,mx^k) = \frac{x^nmx^k}{x^{2n}+m^2x^{2k}} = \frac{mx^{n-k}}{x^{2(n-k)}+m^2} \to 0 \text{ as } x \to 0.$$

But if $k = n$, we have

$$f(x,mx^n) = \frac{m}{1+m^2},$$

so that the limit doesn't exist.

5 Polar coordinates and the squeeze theorem

Many limits may be evaluated by writing the function in polar coordinates and then applying the squeeze theorem. For a limit as $(x,y) \to (0,0)$, we set $x = r\cos\theta, y = r\sin\theta$ and then estimate the result.

Example 6. *Evaluate the following limit if it exists, or show that it does not exist:*

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$

Solution: Let f be the function in the limit. For $r \neq 0$, we have

$$f(r\cos\theta, r\sin\theta) = \frac{r^3\cos^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = r\cos^3\theta.$$

Since the cosine is bounded by 1, we have $|f(r\cos\theta, r\sin\theta)| \leq |r|$. By the squeeze theorem, the limit is 0.

Example 7. *Evaluate the following limit if it exists, or show that it does not exist:*

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^3}{x^2 + y^2}$$

Solution: Let f be the function in the limit. For $r \neq 0$, we have

$$f(r\cos\theta, r\sin\theta) = r^2\cos^4\theta - r\sin^3\theta.$$

To bound this, we will use the triangle inequality ($|x+y| \leq |x| + |y|$): we have

$$|f(r\cos\theta, r\sin\theta)| \leq |r^2\cos^4\theta| + |r\sin^3\theta| \leq r^2 + |r|,$$

so the limit is 0 by the squeeze theorem.

Example 8. *Evaluate the following limit if it exists, or show that it does not exist:*

$$\lim_{(x,y)\to(1,2)} \frac{(x-1)^2(y-2)^2}{(x-1)^2 + (y-2)^2}$$

Solution: Let f be the function in the limit. Note that $(x, y) \to (1, 2)$, so this is not a limit at the origin. We therefore set $x = 1 + r\cos\theta$, $y = 2 + r\sin\theta$: these are polar coordinates centered at $(1, 2)$. For $r \neq 0$, this gives

$$f(1 + r\cos\theta, 2 + r\sin\theta) = r^2\cos^2\theta\sin^2\theta.$$

As above, we estimate

$$|f(1 + r\cos\theta, 2 + r\sin\theta)| \leq r^2,$$

so the limit is 0 by the squeeze theorem.

5.1 The role of θ

In the preceding three examples, we used the boundedness of sin and cos to eliminate the dependency on θ, and concluded that the limit exists using the squeeze theorem. One might wonder if this step is really necessary: is it enough to argue that the right-hand side approaches 0 as $r \to 0$, without first eliminating θ?

Consider Example 4 above: in polar coordinates, we find that for $r \neq 0$,

$$f(r\cos\theta, r\sin\theta) = \frac{r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}.$$

If we now take θ to be fixed, we see that the right-hand side approaches 0 as $r \to 0$. However, *holding θ constant is equivalent to considering lines through the origin*, and we already know that these lines give a limit of 0 for f. On the other hand, we can see that the limit is $\frac{1}{2}$ along the path $y = x^2$, which is $r = \frac{\sin\theta}{\cos^2\theta}$ in polar coordinates (here θ depends on r).

The conclusion is that **we must eliminate the dependence on θ** in order to argue using the squeeze theorem as we did in the preceding examples. (In future math classes, we will say such a bound is *uniform in θ*.)

6 Advanced examples

The next example shows that it is not sufficient to check only polynomial paths through the origin.

Example 9. *Evaluate the following limit if it exists, or show that it does not exist:*

$$\lim_{(x,y)\to(0,0)} \frac{|x|^\alpha y}{|x|^{2\alpha}+y^2},$$

where $\alpha \in (0,1)$ *is irrational.*

Solution: Let f be the function in the limit. Along the path $y = |x|^\alpha$, for $x \neq 0$, we have

$$f(x,|x|^\alpha) = \frac{|x|^\alpha |x|^\alpha}{|x|^{2\alpha}+|x|^{2\alpha}} = \frac{1}{2}.$$

Now consider polynomial paths. If $y = P(x)$, where $P(x) = a_1 x + a_2 x^2 + \ldots + a_n x^n$ is a polynomial with $P(0) = 0$, then for $x \neq 0$, we have

$$P(x)|x|^{-\alpha} = \sum_{j=1}^{n} a_j x^j |x|^{-\alpha} = \sum_{j=1}^{n} a_j (x/|x|)^j |x|^{j-\alpha}.$$

Since $\alpha < 1$, we see that $j > \alpha$ in the sum above. Since $|x/|x|| = 1$, we have $\lim_{x\to 0} P(x)|x|^{-\alpha} = 0$, which shows that

$$f(x,P(x)) = \frac{|x|^\alpha P(x)}{|x|^{2\alpha}+P(x)^2} = \frac{P(x)|x|^{-\alpha}}{1+(P(x)|x|^{-\alpha})^2} \to 0 \text{ as } x \to 0.$$

Now suppose that $x = Q(y)$, where Q is a nonzero polynomial with $Q(0) = 0$. We can write $Q(y) = b_k y^k + \ldots + b_n y^n = y^k(b_k + \ldots + b_n y^{n-k})$ where $b_k \neq 0$. Then

$$\lim_{y\to 0^+} \frac{|Q(y)|^\alpha}{y} = \lim_{y\to 0^+} |y|^{\alpha k} y^{-1} |b_k + \ldots + b_n y^{n-k}|^\alpha,$$

where $|b_k + \ldots + b_n y^{n-k}|^\alpha \to |b_k|^\alpha$ as $y \to 0$. Since α is irrational, we have $\alpha k \neq 1$. Thus we have two cases:

$$\lim_{y\to 0^+} |y|^{\alpha k} y^{-1} = \lim_{y\to 0^+} y^{\alpha k - 1} = \begin{cases} 0 & \alpha k > 1 \\ \infty & \alpha k < 1. \end{cases}$$

This implies that $\lim_{y\to 0^+} |Q(y)|^\alpha / y = 0$ if $\alpha k > 1$ and $\lim_{y\to 0^+} y/|Q(y)|^\alpha = 0$ if $\alpha k < 1$. (By similar reasoning, this latter statement also holds for $y \to 0^-$.)

It follows that

$$f(Q(y),y) = \frac{|Q(y)|^\alpha y}{|Q(y)|^{2\alpha}+y^2} = \frac{|Q(y)|^\alpha / y}{(|Q(y)|^\alpha/y)^2+1} \to 0 \text{ as } y \to 0, \text{ if } \alpha k > 1$$

and

$$f(Q(y),y) = \frac{|Q(y)|^\alpha y}{|Q(y)|^{2\alpha}+y^2} = \frac{y|Q(y)|^{-\alpha}}{1+(y|Q(y)|^{-\alpha})^2} \to 0 \text{ as } y \to 0, \text{ if } \alpha k < 1.$$

Thus, the limit fails to exist even though the limit is 0 along all polynomial paths through the origin.

Example 10. *Evaluate the following limit if it exists, or show that it does not exist:*

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4 + x^2y^2 + y^4}$$

Solution: Let f be the function in the limit. For $r \neq 0$, we have

$$f(r\cos\theta, r\sin\theta) = \frac{r\cos^4\theta\sin\theta}{\cos^4\theta + \cos^2\theta\sin^2\theta + \sin^4\theta}.$$

To obtain an estimate that does not depend on θ, we need to find a *lower* bound for the denominator. We will do this by completing the square. Set $z = \cos^2\theta$. Then the denominator is

$$z^2 + z(1-z) + (1-z)^2 = z^2 - z + 1 = (z - \tfrac{1}{2})^2 + \tfrac{3}{4} \geq \tfrac{3}{4}.$$

It follows that

$$|f(r\cos\theta, r\sin\theta)| = \frac{|r\cos^4\theta\sin\theta|}{\cos^4\theta + \cos^2\theta\sin^2\theta + \sin^4\theta} \leq \frac{|r|}{3/4} \to 0 \text{ as } r \to 0,$$

so the limit is 0 by the squeeze theorem.

Example 11. *Evaluate the following limit if it exists, or show that it does not exist:*

$$\lim_{(x,y)\to(0,0)} \frac{x^5}{x^4 + y^2}$$

Solution: Let f be the function in the limit. For $r \neq 0$, we have

$$f(r\cos\theta, r\sin\theta) = \frac{r^3\cos^5\theta}{r^2\cos^4\theta + \sin^2\theta}.$$

We wish to find a lower bound for the denominator that does not depend on θ. To this end, fix r such that $0 < r < 1$ and set $z = \cos^2\theta \in [0,1]$. We have

$$r^2\cos^4\theta + \sin^2\theta = r^2z^2 + (1-z) =: f(z).$$

We will find a lower bound by minimizing the function. Note that $f'(z) = 2r^2z - 1 < 0$ if $r < 1/\sqrt{2}$. It follows that $f(z) \geq f(1) = r^2$, which shows that

$$r^2\cos^4\theta + \sin^2\theta \geq r^2 \text{ for } 0 < r < 1/\sqrt{2}.$$

Using this estimate, we see that

$$|f(r\cos\theta, r\sin\theta)| \leq |r| \to 0 \text{ as } r \to 0,$$

so the limit is 0 by the squeeze theorem.

Example 12. *Evaluate the following limit if it exists, or show that it does not exist:*

$$\lim_{(x,y)\to(0,0)} \frac{x^4 \sin xy}{x^4 + y^4}$$

Solution: Let f be the function in the limit. For this problem the estimate $|\sin x| \le 1$ will not be sufficient: instead, we will use $|\sin x| \le |x|$, which is a stronger estimate for small x.

For $r \neq 0$, we have

$$f(r\cos\theta, r\sin\theta) = \frac{\cos^4\theta \sin(r^2\cos\theta\sin\theta)}{\cos^4\theta + \sin^4\theta}.$$

As before, we can complete the square to see that

$$\cos^4\theta + \sin^4\theta = \cos^4\theta + (1-\cos^2\theta)^2 = 2(\cos^2\theta - \tfrac{1}{2})^2 + \tfrac{1}{2} \ge \tfrac{1}{2}.$$

Using this and the estimate $|\sin x| \le |x|$, we obtain

$$|f(r\cos\theta, r\sin\theta)| \le 2|\sin(r^2\cos\theta\sin\theta)| \le 2|r^2\cos\theta\sin\theta| \le 2r^2 \to 0 \text{ as } r \to 0,$$

so the limit is 0 by the squeeze theorem.