disc. 9

Power series

$$\sum_{n=0}^{\infty} a_n x^n$$

A power series converges in some interval $(-R, R)$
(R is called the radius of convergence.)
We have $R = \left(\lim_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1}$ if the limit exists.
Note: we use the convention $0^{-1} = \infty$ for this specific situation.

$(-\infty, \infty)$ $(-2, 2)$ $[-1, 1)$

Ex. For which x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Use the ratio test:

$$\left| \frac{x^{n+1}}{(n+1)!} \Big/ \frac{x^n}{n!} \right| = \frac{n!}{(n+1)!} \left| \frac{x^{n+1}}{x^n} \right| = \frac{|x|}{n+1} \xrightarrow{n\to\infty} 0$$

so the series converges for any $x \in \mathbb{R}$.

(Note: can also use root test but it is trickier)

Ex. For which x does $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converge?

Use the root test:

$$\sqrt[n]{\left|\frac{x^n}{n}\right|} = \frac{|x|}{\sqrt[n]{n}} \xrightarrow{n\to\infty} |x|$$

if $|x| < 1$, series converges
if $|x| > 1$, series diverges
if $|x| = 1$, test is inconclusive

$\left(\ln(\sqrt[n]{n}) = \frac{1}{n} \ln n \to 0 \quad \text{so} \quad \sqrt[n]{n} \to 1\right)$

So the series converges in $(-1, 1)$. Still need to check edges.

$x = 1$: $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test

So the interval of convergence is $[-1, 1)$.

Suppose $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ in some $(-R+a, R+a)$.

Since we can differentiate power series term-by-term, we have

$$f'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n (x-a)^n$$

$$= \sum_{n=1}^{\infty} n a_n (x-a)^{n-1} \qquad (|x-a| < R)$$

Similarly,

$$f''(x) = \sum_{n=1}^{\infty} \frac{d}{dx}\left(n a_n (x-a)^{n-1}\right) = \sum_{n=2}^{\infty} n(n-1) a_n (x-a)^{n-2}$$

$\dots$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n (x-a)^{n-k}$$

Evaluate this at $x = a$: we get $f^{(k)}(a) = \underbrace{k(k-1) \cdots 1}_{k!} \cdot a_k$

$$\Rightarrow \quad a_k = \frac{f^{(k)}(a)}{k!}$$

(Warning: the power series for f may converge to f only at a single point:

e.g. $f(x) = \begin{cases} e^{-1/|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$

This f is infinitely differentiable and $f^{(k)}(0) = 0$ for all k.

So the power series for f about $x = 0$ is $\sum_{k=0}^{\infty} \frac{0}{k!} x^k \equiv 0$.

The series only converges to f at $x = 0$.)

<u>Ex.</u> $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x everywhere

<u>Ex.</u> $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ in $|x| < 1$.

Ex. Consider the function $f(x) = \frac{1}{1-x}$. We would like to find the associated power series about $a = 0, 2$.

Let's find the coefficients using the formula $a_n = \frac{f^{(n)}(a)}{n!}$.

What is $f^{(n)}(a)$?

$$\frac{d}{dx}\frac{1}{1-x} = -(1-x)^{-2}(-1) = (1-x)^{-2}$$

$$\frac{d^2}{dx^2}\frac{1}{1-x} = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

$$\frac{d^3}{dx^3}\frac{1}{1-x} = 2(-3)(1-x)^{-4}(-1) = 6(1-x)^{-4}$$

$$\cdots$$

$$\frac{d^n}{dx^n}\frac{1}{1-x} = n!\,(1-x)^{-(n+1)}$$

Series about $a = 0$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{\cancel{n!}}{(1-0)^{n+1}} \frac{1}{\cancel{n!}} x^n = \sum_{n=0}^{\infty} x^n \qquad (|x| < 1)$$

Series about $a = 2$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \sum_{n=0}^{\infty} \frac{\cancel{n!}}{(1-2)^{n+1}} \frac{1}{\cancel{n!}} (x-2)^n = \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n$$

$$(|x-2| < 1)$$

[Q: does the second series converge to f everywhere in $|x-2| < 1$?

Note that $\sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n = -\sum_{n=0}^{\infty} (2-x)^n$ is a geometric series

so this converges, for $|x-2| < 1$, to $\frac{-1}{1-(2-x)} = \frac{-1}{-1+x} = \frac{1}{1-x}$

so the series indeed converges to f.]

Recall the series for e^x, $\cos x$, $\sin x$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$2n \quad n \geq 0 \qquad 2n+1 \quad n \geq 0$$

We claim: $e^{ix} = \cos x + i \sin x$

Indeed,

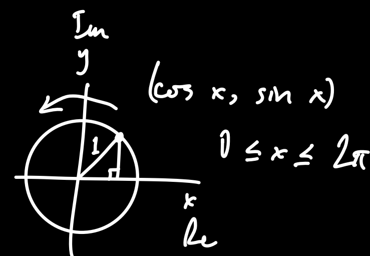
$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \underbrace{\sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!}}_{\text{even powers of } x} + \underbrace{\sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!}}_{\text{odd powers of } x}$$

$$= \sum_{n=0}^{\infty} \frac{i^{2n} x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1} x^{2n+1}}{(2n+1)!}$$

$i^{2n} = (i^2)^n = (-1)^n$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \cos x + i \sin x$$



$(x, y) \longleftrightarrow x + iy$

$\mathbb{R}^2 \longleftrightarrow \mathbb{C}$

$e^{ix} = \cos x + i \sin x$

So e^{ix} traces out the unit circle CCW

(for $0 \leq x \leq 2\pi$)

Also, $e^{i\pi} = \cos \pi + i \cancel{\sin \pi}^{\,0} = -1$

disc. 9 Power series

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

A power series converges in some interval $(-R+a, R+a)$ where R is called the radius of convergence.

If the limit exists, we have $R = \left(\lim_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1}$ [where we say $0^{-1} = \infty$]

Ex. For which x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Use ratio test:

$$\left| \frac{x^{n+1}}{(n+1)!} \Big/ \frac{x^n}{n!} \right| = \left| \frac{x^{n+1}}{x^n} \right| \frac{n!}{(n+1)!} = \frac{|x|}{n+1} \xrightarrow{n\to\infty} 0$$

So the series converges for all x.

Note: can also use the root test, but it is trickier.

Ex. For which x does $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converge?

Use root test:

$$\sqrt[n]{\left|\frac{x^n}{n}\right|} = \frac{|x|}{\sqrt[n]{n}} \xrightarrow{n\to\infty} |x|$$

if $|x| < 1$, series converges
if $|x| > 1$, series diverges
if $|x| = 1$, test is inconclusive

Conv. in $(-1, 1)$.

Check edges:

$x = 1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic series $\rightarrow$ diverges.

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test.

So we conclude: the interval of convergence is $[-1, 1)$.

Suppose $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ in $(-R+a, R+a)$.

Since we can differentiate power series term-by-term, we have

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n (x-a)^n$$

$$= \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

Similarly,

$$f''(x) = \sum_{n=1}^{\infty} \frac{d}{dx} n a_n (x-a)^{n-1} = \sum_{n=2}^{\infty} n(n-1) a_n (x-a)^{n-2}$$

...

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)\, a_n (x-a)^{n-k}$$

Evaluate at $x = a$:

$$f^{(n)}(a) = n(n-1)(n-2)\cdots 1 \cdot a_n \implies a_n = \frac{f^{(n)}(a)}{n!}.$$

(Warning: the power series for f may converge at only one point:

e.g., $f(x) = \begin{cases} e^{-1/|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$

This f is infinitely differentiable with $f^{(n)}(0) = 0$ for all n.

So the power series for f about $x = 0$ is $\sum_{n=0}^{\infty} \frac{0}{n!} x^n \equiv 0$.

Thus the series only converges to f at $x = 0$.)

<u>Ex.</u> $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x everywhere

<u>Ex.</u> $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ if $|x| < 1$.

Consider the function $f(x) = \frac{1}{1-x}$. We would like to find the associated power series about $a = 0,\ 2,\ 1/2$.

What is $f^{(n)}(x)$?

$$\frac{d}{dx}\frac{1}{1-x} = -((1-x)^{-2}(-1)) = (1-x)^{-2}$$

$$\frac{d^2}{dx^2}\frac{1}{1-x} = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

$$\frac{d^3}{dx^3}\frac{1}{1-x} = -3\cdot 2(1-x)^{-4}(-1) = 6(1-x)^{-4}$$

...

$$\frac{d^n}{dx^n}\frac{1}{1-x} = n!(1-x)^{-(n+1)}$$

Series about $a = 0$:

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{\cancel{n!}}{(1-0)^{n+1}}\cdot\frac{x^n}{\cancel{n!}} = \sum_{n=0}^{\infty}x^n \qquad \text{conv. in } |x|<1$$

Series about $a = 2$:

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty}\frac{\cancel{n!}}{(1-2)^{n+1}}\frac{(x-2)^n}{\cancel{n!}} = \sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n \qquad \text{conv. in } |x-2|<1$$

Note that this converges to $\frac{1}{1-x}$:

$$\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n = -\sum_{n=0}^{\infty}(2-x)^n$$

$$\overset{|x-2|<1}{=} \frac{-1}{1-(2-x)}$$

$$= \frac{-1}{-1+x} = \frac{1}{1-x}$$

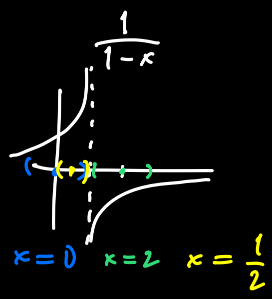
Series about $a = 1/2$:

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(1/2)}{n!}\left(x-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}\frac{\cancel{n!}}{\left(1-\frac{1}{2}\right)^{n+1}}\cdot\frac{\left(x-\frac{1}{2}\right)^n}{\cancel{n!}}$$

$$= \sum_{n=0}^{\infty} 2^{n+1}\left(x - \tfrac{1}{2}\right)^{n} = 2\sum_{n=0}^{\infty} (2x-1)^{n}$$

conv. if $|2x - 1| < 1$

i.e. if $\left|x - \tfrac{1}{2}\right| < \tfrac{1}{2}$



disc. 10

- Differentiation of series

Ex. $\sum_{n=1}^{\infty} \frac{n}{2^n} = x\sum_{n=1}^{\infty} n x^{n-1} \Big|_{x=1/2}$

$= \frac{x}{(1-x)^2}\Big|_{x=1/2}$

$= \frac{1/2}{(1-1/2)^2}$

$= 2$

Recall: we can differentiate power series term-by-term.
If $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges in $(-R+a, R+a)$,
we have $\frac{d}{dx}\sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$
in $(-R+a, R+a)$.

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\frac{1}{1-x}$$

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \qquad |x|<1$$

$$\left(\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n\right)$$

Ex. $\sum_{n=1}^{\infty} \frac{n^2}{\pi^n} \overset{x=1/\pi}{=} \frac{2/\pi^2}{(1-1/\pi)^3} + \frac{1/\pi}{(1-1/\pi)^2}$

$$\sum_{n=1}^{\infty} n^2 x^n = \sum_{n=1}^{\infty} (n(n-1)+n)x^n$$

$$= x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} n x^{n-1}$$

$$= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \qquad (|x|<1)$$

$$\frac{d^2}{dx^2}\sum_{n=0}^{\infty} x^n = \frac{d^2}{dx^2}\frac{1}{1-x} = \frac{2}{(1-x)^3}$$

$$= \frac{d}{dx}\sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} \qquad |x|<1$$

$$-\sum_{n=0}^{\infty} (2-x)^n = \frac{1}{1-x} \qquad |x-2|<1$$

Probability example We toss a coin repeatedly w/ prob. of heads p, $0<p<1$.
What is the expected $\underbrace{\text{number of tries until the first heads}}_{=:X}$?

$$\underbrace{TT\cdots T}_{n-1}H$$

$$P(X=n) = (1-p)^{n-1}p$$

$$\mathbb{E}X = \sum_{n=1}^{\infty} nP(X=n) = \sum_{n=1}^{\infty} n(1-p)^{n-1}p = \frac{p}{(1-(1-p))^2} = \frac{1}{p}$$

From above: $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$

$\underline{e}$

$e = 2.718281\ldots$ $\qquad$ $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \overset{x=1}{\Longrightarrow} \quad e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

<u>Proposition</u> e is irrational.

<u>Pf.</u> cf. Rudin (Principles)

Let $e_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$.

Then we have

$$\begin{aligned}
0 < e - e_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\
&= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\
&= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right) \\
&< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) \\
&= \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \left(\frac{1}{n+1}\right)^k \\
&= \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+1}} \\
&= \frac{1}{n!\,n}
\end{aligned}$$

So we have a bound $0 < e - e_n < \dfrac{1}{n!\,n}$.

Suppose e is rational, i.e. $e = \dfrac{p}{q}$ where p, q are positive integers.

Then we have $0 < e - e_q < \dfrac{1}{q!\,q}$, so that

$$0 < q!(e - e_q) < \frac{1}{q} \leq 1$$

Now, note that $q!e = q!\frac{p}{q} = p(q-1)!$ is an integer

Moreover, $q!e_q = q!\left(1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right)$

$$= q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{q!}$$ is also an integer.

Therefore $q!(e - e_q)$ is an integer.

But we showed above that it is in $(0, 1)$, which is a contradiction. $\lightning$