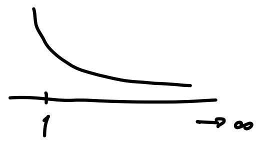
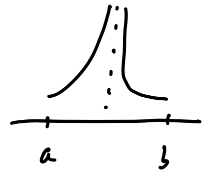
disc. 6

Improper integrals





$$\int_0^1 \ln x \, dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \ln x \, dx$$
$$= \lim_{\varepsilon \to 0^+} (x \ln x - x)\Big|_\varepsilon^1$$
$$= \lim_{\varepsilon \to 0^+} (1 \cancel{\ln 1}^{\,0} - 1) - (\varepsilon \ln \varepsilon - \varepsilon) = -1$$

where $\displaystyle \lim_{\varepsilon \to 0^+} \varepsilon \ln \varepsilon = \lim_{\varepsilon \to 0^+} \frac{\ln \varepsilon}{1/\varepsilon} \overset{L'H}{=} \lim_{\varepsilon \to 0^+} \frac{1/\varepsilon}{-1/\varepsilon^2} = \lim_{\varepsilon \to 0^+} (-\varepsilon) = 0.$

$$\int_1^\infty \frac{dx}{x^3} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^3} = \lim_{b \to \infty} \left(-\frac{1}{2} x^{-2}\right)\Big|_1^b = \lim_{b \to \infty} \left(-\frac{1}{2} b^{-2} - \left(-\frac{1}{2}\right)\right) = \frac{1}{2}.$$

Induction

Suppose $P(n)$ is a statement involving positive integers $n \in \mathbb{Z}^+$.
If (i) $P(1)$ is true (base case)
& (ii) $P(n) \Rightarrow P(n+1)$ holds (induction step)
then $P(n)$ is true for all n.

$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow P(4) \Rightarrow \dots$

Ex. $P(n) =$ "$1 + \dots + n = \frac{n(n+1)}{2}$"

Check base case: $1 = \frac{1(1+1)}{2}$ ✓

Induction step holds: suppose that $P(n)$ is true for some n: i.e., $1 + \dots + n = \frac{n(n+1)}{2}$.

Then $1 + \dots + n + (n+1) = (1 + \dots + n) + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$.

so $P(n+1)$ holds.

Sequences

$a_n : \mathbb{Z}^+ \to \mathbb{C}$

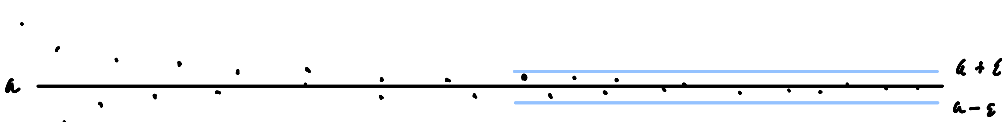
$a_1, a_2, a_3, a_4, a_5, \dots$

Note: $\lim_{x \to \infty} f(x) = A \Rightarrow \lim_{n \to \infty} f(n) = A$
but not vice versa, ($\not\Leftarrow$)
e.g. $f(x) = \cos 2\pi x$.
Then $\lim_{n \to \infty} f(n) = 1$ but $\lim_{x \to \infty} f(x)$ DNE

Defn. of limit

$$\lim_{n \to \infty} \frac{n^2}{e^n} = 0$$

We say $\lim_{n \to \infty} a_n = a$ if:
For all $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $n > N \Rightarrow |a_n - a| < \varepsilon$.



Ex. $a_1 = 1, \quad a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right)$

Does this seq. converge? If so, what is the limit?

Prop. Suppose b_n is increasing & bounded above or decreasing & bounded below. Then b_n has a limit.

Solution

We claim:

(i) $a_n^2 \geq 2$ for $n \geq 2$ $\quad (a_n \geq \sqrt{2})$

(ii) $a_{n+1} \leq a_n$ for $n \geq 2$

First check (i).

Base case: $a_2^2 = \left(\frac{1}{2}\left(a_1 + \frac{2}{a_1}\right)\right)^2 = \left(\frac{1}{2}\cdot(1+2)\right)^2 = \frac{9}{4} \geq 2.$ ✓

Induction step: Suppose $a_n^2 \geq 2$ for some n.

$$\text{Then } a_{n+1}^2 - 2 = \left(\frac{1}{2}\left(a_n + \frac{2}{a_n}\right)\right)^2 - 2$$

$$= \frac{1}{4}\left(a_n^2 + \frac{4}{a_n^2} + 2\cancel{a_n}\cdot\frac{2}{\cancel{a_n}}\right) - 2$$

$$= \frac{a_n^2}{4} + \frac{1}{a_n^2} - 1$$

$$= \left(\frac{a_n}{2} - \frac{1}{a_n}\right)^2 \geq 0$$

that is, $a_{n+1}^2 \geq 2$.

Thus (i) holds.

Now check (ii): $a_{n+1} \leq a_n$ is equivalent to

$\frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \leq a_n$

$\Leftrightarrow \quad a_n + \frac{2}{a_n} \leq 2a_n$

$\Leftrightarrow \quad \frac{2}{a_n} \leq a_n$

$\Leftrightarrow \quad 2 \leq a_n^2$ which is (i). Thus (ii) holds.

Thus, using the Prop. above, we see that a_n converges. Let's say $\lim_{n\to\infty} a_n = x$.

In particular, $\lim_{n\to\infty} a_{n-1} = x$.

$a_n \geq 0$

$x = \lim_{n\to\infty} a_n \geq 0$

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right)$$

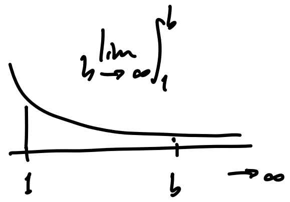
As $n \to \infty$, we get

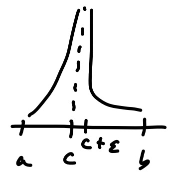
$$x = \frac{1}{2}\left(x + \frac{2}{x}\right) = \frac{x}{2} + \frac{1}{x}$$

$$\Rightarrow \frac{x}{2} = \frac{1}{x} \Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}.$$

disc. 6

Improper integrals





$\lim_{\varepsilon\to0^+}\int_\varepsilon^b f$

$$\int_0^1 \ln x\,dx = \lim_{\varepsilon\to0^+}\int_\varepsilon^1 \ln x\,dx$$
$$= \lim_{\varepsilon\to0^+}(x\ln x - x)\Big|_\varepsilon^1$$
$$= \lim_{\varepsilon\to0^+}(1\cancel{\ln 1}^{\,0} - 1) - (\varepsilon\ln\varepsilon - \varepsilon)$$
$$= -1 \quad \text{using L'H to see } \lim_{\varepsilon\to0^+}\varepsilon\ln\varepsilon = 0$$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p} = \begin{cases} \lim_{b\to\infty}(\ln b - \ln 1) = +\infty & p = 1 \\ \lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}\right) & p \neq 1 \end{cases}$$

$$\frac{x^{-p+1}}{-p+1}\Big|_1^b$$

$$\to \begin{cases} +\infty & p < 1 \\ \frac{1}{p-1} & p > 1 \end{cases}$$

Induction

Suppose $P(n)$ is a statement depending on a positive integer n.
If (i) $P(1)$ is true (base case)
& (ii) $P(n) \Rightarrow P(n+1)$ for all n (induction step)
then $P(n)$ is true for all n.

$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow P(4) \Rightarrow \dots$

Ex. $P(n)$ = "$1 + \dots + n = \frac{n(n+1)}{2}$"

(i) $P(1)$: $1 = \frac{1(1+1)}{2}$ ✓

(ii) Suppose $P(n)$ is true for some n. Want to show $P(n+1)$ is true.

$$1 + \dots + (n+1) = (1 + \dots + n) + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2}$$

which is $P(n+1)$.

Sequences

$a_n : \mathbb{Z}^+ \to \mathbb{C}$

$a_1, a_2, a_3, a_4, a_5, \dots$

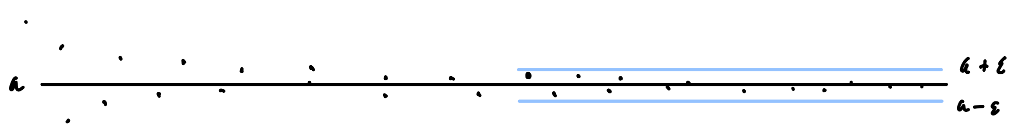
Note: If $\lim_{x\to\infty} f(x) = A$, then $\lim_{n\to\infty} f(n) = A$.
Other direction is false. E.g.:
$f(x) = \cos 2\pi x$. Then $f(n) = \cos 2\pi n = 1$
for $n \in \mathbb{Z}^+$; $\lim_{n\to\infty} f(n) = 1$, but $\lim_{x\to\infty} f(x)$ DNE.

$$\lim_{n\to\infty}\frac{n^2}{e^n} = \lim_{x\to\infty}\frac{x^2}{e^x} = 0$$

Defn. of limit

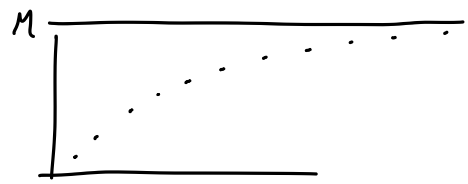
We say $a_n \to a$, or $\lim_{n\to\infty} a_n = a$, if:

For all $\varepsilon > 0$, there exists a positive integer N such that: $n > N \Rightarrow |a_n - a| < \varepsilon$.



Ex. Suppose $c \geq 1$, $a_1 = \sqrt{c}$, $a_n = \sqrt{c\,a_{n-1}}$ for $n \geq 2$.
Does this seq. converge? If so, what is the limit?

Prop. Suppose b_n is increasing & bounded above or decreasing & bounded below. Then b_n has a limit.



$b_n \to c \leq M$

Claim: $a_n \leq c$ for all n.

Base case: $a_1 = \sqrt{c} \leq c$ ✓

Induction step: suppose $a_{n-1} \leq c$.

Then $a_n = \sqrt{c\,a_{n-1}} = \sqrt{c}\sqrt{a_{n-1}} \leq \sqrt{c}\sqrt{c} = c$.

So the claim is true by induction

Also claim, sequence is increasing:

$a_n \geq a_{n-1}$ ✓

$$a_n = \sqrt{\underset{\substack{\uparrow \\ c \geq a_{n-1}}}{c}\,a_{n-1}} \geq \sqrt{a_{n-1}^2} = a_{n-1}$$

Thus the limit exists: say $\lim_{n\to\infty} a_n = x$.

$\lim_{n\to\infty} a_{n-1} = x$

Let $n \to \infty$ in $\underset{\substack{\downarrow \\ x}}{a_n} = \sqrt{c\,\underbrace{a_{n-1}}_{\substack{\downarrow \\ x}}}$:

$$x = \sqrt{cx} \Rightarrow x^2 = cx \Rightarrow x = \cancel{0}, c$$

So the limit is c.

disc. 7

Series Suppose a_n is a sequence of real numbers. We say the series $\sum a_n$ converges to s if $\lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} a_n}_{=:S_N} = s.$ $\qquad \lim_{N\to\infty}(a_1 + \cdots + a_N)$

$=: S_N$ is the nth partial sum of the series $\qquad \sum_{n=1}^{\infty}(-1)^n \qquad -1+1-1+1-\ldots$

$-1, 0, -1, 0, \ldots$

Geometric series

We have

$$\sum_{n=0}^{\infty} r^n \begin{cases} = \dfrac{1}{1-r} & \text{if } |r|<1 \\ \text{div.} & \text{if } |r| \geq 1 \end{cases} \qquad \sum_{n=2}^{\infty} r^n = \frac{1}{1-r} - 1 - r \quad \text{etc.}$$

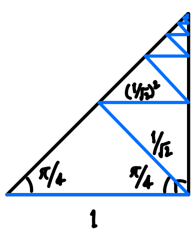
Pf. Write $S_N = 1 + r + r^2 + \cdots + r^N$. Then

$$(1-r)S_N = 1 + \cancel{r} + \cancel{r^2} + \cdots + \cancel{r^N} - \cancel{r} - \cancel{r^2} - r^3 - \cdots - \cancel{r^N} - r^{N+1}$$

$$= 1 - r^{N+1}$$

$$\Rightarrow S_N = \frac{1-r^{N+1}}{1-r} \longrightarrow \frac{1}{1-r} \quad \text{if } |r|<1$$

59.



What is the length of the blue path?

$$L = \sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{1-1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}-1}.$$

$$\left(\frac{1}{\sqrt{2}}\right)^0 + \left(\frac{1}{\sqrt{2}}\right)^1 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cdots$$

The divergence test

Prop. $\sum_{n=1}^{\infty} a_n$ converges $\overset{\not\Leftarrow}{\Rightarrow}$ $a_n \to 0$

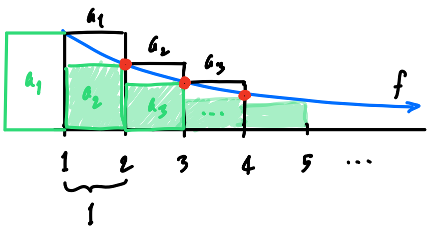
e.g. $\sum_{1}^{\infty}\frac{1}{n} = \infty, \quad \sum_{1}^{\infty}\frac{1}{n^2} < \infty$

Methods for nonnegative series

The integral test

If $a_n = f(n)$, where f is nonneg., decreasing, [continuous] for $x \geq 1$, then

$$\int_1^\infty f(x)\,dx \quad \text{converges iff.} \quad \sum_{n=1}^\infty a_n \quad \text{converges}$$



$$\sum_{n=2}^\infty a_n \leq \int_1^\infty f(x)\,dx \leq \sum_{n=1}^\infty a_n$$
$$= a_1 + \sum_{n=2}^\infty a_n \leq a_1 + \int_1^\infty f(x)\,dx$$

11. $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^{3/2}}$ compare to $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^{3/2}} \underset{du = \frac{dx}{x}}{\overset{u = \ln x}{=}} \int_{\ln 2}^\infty \frac{du}{u^{3/2}} = -2u^{-1/2}\Big|_{\ln 2}^\infty$
$= 2(\ln 2)^{-1/2}$

so this converges, hence the series converges

Ex. $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ $\begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$ $\left(\text{look at } \displaystyle\int_1^\infty \frac{dx}{x^p}\right)$

The direct comparison test

If $0 \leq a_n \leq b_n$ for sufficiently large n, then

(i) $\sum b_n < \infty \Rightarrow \sum a_n < \infty$

(ii) $\sum a_n = \infty \Rightarrow \sum b_n = \infty$

Ex. $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+1}$ $\qquad \displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$

$a_n = \dfrac{1}{n^2+1}$ $\qquad b_n = \dfrac{1}{n^2}$ $\qquad \sum b_n < \infty \overset{DCT}{\Rightarrow} \sum a_n < \infty$

$a_n \leq b_n$

$\Leftrightarrow \dfrac{1}{n^2+1} \leq \dfrac{1}{n^2}$

$\Leftrightarrow n^2 \leq n^2 + 1$

$1 - \varepsilon \leq \dfrac{n^2}{n^2-1} \leq 1 + \varepsilon$

e.g. $\varepsilon = 1$:

$n^2 \leq 2(n^2 - 1)$

$\Downarrow$

$\dfrac{1}{n^2-1} \leq \dfrac{2}{n^2}$

Ex. $\displaystyle\sum_{n=2}^\infty \frac{1}{n^2-1}$ $\qquad \displaystyle\sum_{n=2}^\infty \frac{2}{n^2}$ $\qquad a_n = \dfrac{1}{n^2-1}$ $\qquad b_n = \dfrac{2}{n^2}$ $\qquad a_n \leq b_n$ ✓

$\dfrac{1}{n^2-1} \leq \dfrac{1}{n^2} \Leftrightarrow \cancel{n^2 \leq n^2 - 1}$ $\qquad \dfrac{n^2}{n^2-1} \longrightarrow 1$

Then $\sum \dfrac{1}{n^2-1}$ converges by DCT

59. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^4}$

67. $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n}$

68. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{\sqrt{n}}$

70. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{e^{\sqrt{n}}}$

72. $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

disc. 7

Series Suppose a_n is a sequence of real numbers. We say the series $\sum a_n$ converges to s if
$$\lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} a_n}_{=:\, S_N} = s. \qquad \lim_{N\to\infty}(a_1 + \cdots + a_N) = s$$
$=: S_N$ is the nth partial sum of the series

Geometric series

We have
$$\sum_{n=0}^{\infty} r^n \begin{cases} = \dfrac{1}{1-r} & \text{if } |r|<1 \\ \text{div.} & \text{if } |r| \geq 1 \end{cases}$$

Pf. Write $S_N = 1 + r + r^2 + \cdots + r^N$. Then
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$$= 1 - r^{N+1}$$
$$\Rightarrow S_N = \frac{1-r^{N+1}}{1-r} \longrightarrow \frac{1}{1-r} \quad \text{if } |r|<1$$

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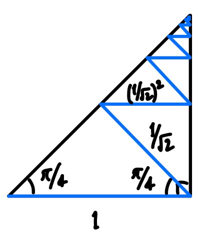

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$$\left(\frac{1}{\sqrt{2}}\right)^0 + \left(\frac{1}{\sqrt{2}}\right)^1 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cdots$$

The divergence test

Prop. $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow$ $a_n \to 0$ ($\cancel{\Leftarrow}$)

$$\sum_{n=1}^{\infty}\frac{1}{n} = \infty \qquad \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$$

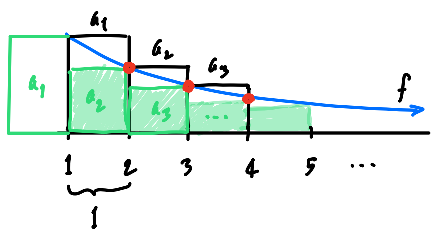
$$\sum_{n=1}^{\infty} 3^n \cos n$$

Methods for nonnegative series

The integral test

If $a_n = f(n)$, where f is nonneg., decreasing, [continuous] for $x \geq 1$, then

$$\int_1^\infty f(x)\,dx \text{ converges iff. } \sum_{n=1}^\infty a_n \text{ converges}$$



$$\sum_{n=2}^\infty a_n \leq \int_1^\infty f(x)\,dx \leq \sum_{n=1}^\infty a_n$$

11. $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^{3/2}}$ compare to $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^{3/2}} \underset{du = \frac{dx}{x}}{\overset{u = \ln x}{=}} \int_{\ln 2}^\infty \frac{du}{u^{3/2}} = -2u^{-1/2}\Big|_{\ln 2}^{\infty} = 2(\ln 2)^{-1/2}$

so the sum converges.

Ex. $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ $\begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$ $\left(\text{compare to } \displaystyle\int_1^\infty \frac{dx}{x^p}\right)$

The direct comparison test

If $0 \leq a_n \leq b_n$ for sufficiently large n, then

(i) $\sum b_n < \infty \Rightarrow \sum a_n < \infty$ $\qquad \cancel{\sum b_n < \infty}$

(ii) $\sum a_n = \infty \Rightarrow \sum b_n = \infty$ $\qquad \cancel{\sum a_n < \infty}$

Ex. $\displaystyle\sum_{n=1}^\infty \underbrace{\frac{1}{n^2+1}}_{a_n}$ $\qquad \displaystyle\sum_{n=1}^\infty \underbrace{\frac{1}{n^2}}_{b_n}$ conv. $\qquad$ By comparison, $\sum a_n$ converges.

$a_n \leq b_n$

$\Leftrightarrow \dfrac{1}{n^2+1} \leq \dfrac{1}{n^2} \Leftrightarrow n^2 \leq n^2 + 1$ ✓

$$1 - \varepsilon \leq \frac{n^2}{n^2-1} \leq 1 + \varepsilon$$

$$\frac{1}{n^2-1} \Big/ \frac{1}{n^2} = \frac{n^2}{n^2-1} \longrightarrow 1$$

Ex. $\displaystyle\sum_{n=2}^\infty \frac{1}{n^2-1}$ $\qquad \displaystyle\sum_{n=2}^\infty \frac{2}{n^2}$ conv. $\qquad$ By comparison, $\displaystyle\sum \frac{1}{n^2-1}$ converges.

$$\frac{1}{n^2-1} \leq \frac{2}{n^2} \Leftrightarrow n^2 \leq 2(n^2-1) \Leftrightarrow 2 \leq n^2$$

59. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^4}$ $\qquad \displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ div.

For any $\alpha > 0$:

$\ln n \le n^{\alpha}$ eventually

$n^{\alpha} \le e^{n}$ eventually

$$\frac{1}{(\ln n)^4} \ge \frac{1}{n} \iff n^{1/4} \ge \ln n$$

by comparison, the series diverges

$n = \ln k$

$(\ln k)^{\alpha} \le k \qquad x^{\alpha} \le e^{x}$

$$\frac{x^{\lceil \alpha \rceil}}{e^{x}} \le 1$$

62. $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$ $\qquad \displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ div.

$\downarrow$

series diverges.

I. $-\frac{\pi}{2} < x < \pi/2$

$$\cos x \le \frac{\sin x}{x} \le 1$$

$$x \cos x \le \sin x$$

$$\frac{1}{2n} \le \frac{1}{n} \cos\frac{1}{n} \le \sin\frac{1}{n}$$

II. $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$

$$\frac{\sin 1/n}{1/n} \longrightarrow 1 \leftarrow 0 < L < \infty$$

$\displaystyle\sum_n \sin\frac{1}{n}$, $\displaystyle\sum_n \frac{1}{n}$ converge or diverge simultaneously by limit comparison

68. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{\sqrt{n}}$ vs. $\displaystyle\sum_n \frac{1}{n^{3/2}}$

I. $\dfrac{\sin x}{x} \le 1$

$$\frac{\sin 1/n}{\sqrt{n}} \le \frac{1}{n^{3/2}}$$

converges by direct comparison

II. $\dfrac{\sin x}{x} \to 1$ So, the series diverges

$$\frac{\sin 1/n}{\sqrt{n}} \Big/ \frac{1}{n^{3/2}} = \frac{\sin 1/n}{1/n} \longrightarrow 1$$

(then conclude by limit comparison)

70. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{e^{\sqrt{n}}}$ vs. $\displaystyle\sum_n \frac{1}{n^2}$

$\downarrow$

converges.

$$\frac{1}{e^{\sqrt{n}}} \le \frac{1}{n^2}$$

$$\iff n^2 \le e^{\sqrt{n}}$$

$$\underset{k=\sqrt{n}}{\iff} k^4 \le e^{k}$$

72. $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

$\downarrow$

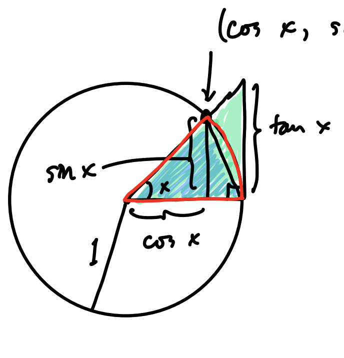
diverges by lim. comparison.

$$\left(1 + \frac{1}{n}\right)^{n} \longrightarrow e$$

$$n \ln\left(1 + \frac{1}{n}\right) \longrightarrow 1$$

$$\Rightarrow \frac{\ln\left(1 + 1/n\right)}{1/n} \longrightarrow 1$$

use limit comparison vs. $\displaystyle\sum \frac{1}{n}$



$0 \le x < \pi/2$

blue: area $\frac{1}{2} \sin x$

red sector: area $\frac{x}{2}$

green: area $\frac{1}{2} \tan x$

$$\cos x \le \frac{\sin x}{x} \le 1$$

$$\cancel{\tfrac{1}{2}} \sin x \le \cancel{\tfrac{1}{2}} x \le \cancel{\tfrac{1}{2}} \tan x$$

sin/cos

22 Feb.

For $\alpha > 0$: $\ln n \le n^\alpha$ for large n

$n^\alpha \le e^n$ for large n

59. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^4}$: vs. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$

$\frac{1}{(\ln n)^4} \overset{?}{\geq} \frac{1}{n} \iff n \geq (\ln n)^4 \iff n^{1/4} \geq \ln n$

so, diverges by comparison test

67. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\sin \frac{1}{n}}_{= a_n}$: vs. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$

I. Direct bound

$\underbrace{\cos x \le \frac{\sin x}{x}} \le 1$

$x \cos x \le \sin x$

$\frac{1}{2n} \le \frac{1}{n} \cos \frac{1}{n} \le \sin \frac{1}{n}$

(n large enough that $\cos \frac{1}{n} \geq \frac{1}{2}$)

so series diverges by direct comparison

II. Limit comparison

Compare to $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{n}}_{= b_n}$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\sin 1/n}{1/n} = 1$

$0 < L < \infty$

So, since $\sum \frac{1}{n}$ diverges, limit comparison shows that $\sum \sin \frac{1}{n}$ also diverges

68. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{\sin \frac{1}{n}}{\sqrt{n}}}_{= a_n}$ vs. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^{3/2}}}_{= b_n}$

$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{(\sin 1/n)/\cancel{\sqrt{n}}}{1/n \cdot 1/\cancel{\sqrt{n}}} = 1 \qquad 0 < L < \infty$

By limit comparison, $\sum \frac{\sin 1/n}{\sqrt{n}}$ converges.

70. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{e^{\sqrt{n}}}$ vs. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2}$

$n^{\alpha/2} \le e^{\sqrt{n}}$ for large n

$\frac{1}{e^{\sqrt{n}}} \overset{?}{\le} \frac{1}{n^2} \iff n^2 \le e^{\sqrt{n}}$ ✓

So the series converges by direct comparison

72. $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

$\left(1 + \frac{1}{n}\right)^n \to e$

$\ln(1 + x) = x + (\ldots)$

$n \ln\left(1 + \frac{1}{n}\right) \to 1$

$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$

$\frac{a_n}{b_n} = \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} \to 1 \qquad 0 < L < \infty$

So by limit comparison, original series diverges.

Absolutely & conditionally convergent series

We say $\sum a_n$ <u>converges absolutely</u> if $\sum |a_n|$ converges. (Then $\sum a_n$ converges as well)

We say $\sum a_n$ <u>converges conditionally</u> if $\sum a_n$ converges but $\sum |a_n|$ diverges.

↳ e.g. $\sum_{1}^{\infty} \frac{(-1)^n}{n}$

Alternating series test

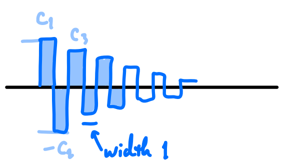
The series $\sum_{n=1}^{\infty} (-1)^{n+1} c_n$ converges if $c_n \geq 0$, c_n decreases, and $c_n \to 0$.

Determine whether the series conv. absolutely, conditionally, or not at all:

$\sum_{1}^{\infty} \frac{\sin(en)}{n^2}$ $\qquad \left|\frac{\sin en}{n^2}\right| \leq \frac{1}{n^2}$

By direct comparison, $\sum \left|\frac{\sin en}{n^2}\right|$ converges.

→ so this converges absolutely.



$c_1 - c_2 + c_3 - c_4 + c_5 - \cdots$

9. $\sum_{2}^{\infty} \frac{(-1)^n}{n \ln n}$

Consider $\sum_{2}^{\infty} \left|\frac{(-1)^n}{n \ln n}\right|$ compare to $\int_2^{\infty} \frac{dx}{x \ln x} \overset{u = \ln x}{\underset{du = \frac{dx}{x}}{=}} \int_{\ln 2}^{\infty} \frac{du}{u} = \ln \infty - \ln \ln 2 = \infty$

↳ diverges by integral test. So, series does not converge absolutely.

To see if the series converges, use alternating series test.

- $\frac{1}{n \ln n} \geq 0$ ✓
- $\frac{1}{n \ln n} \to 0$ ✓
- $\frac{1}{n \ln n}$ decreasing: → Consider $f(x) = \frac{1}{x \ln x}$:

For $x \geq 2$: $f'(x) = -(x \ln x)^{-2}(\ln x + 1)$

$\leq -\frac{1}{(x \ln x)^2} \leq 0$

so f is decreasing ✓

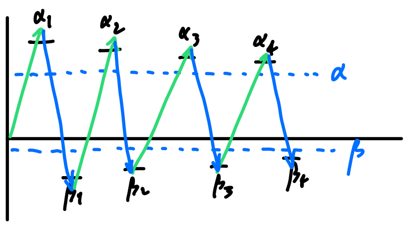
So by alternating series test, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges. Hence the series converges conditionally.

Rearrangement of series

Given a_n, a <u>rearrangement</u> is $a_{\sigma(n)}$ where σ is a <u>permutation</u>,

i.e. $\sigma : \mathbb{N} \circlearrowleft$ is bijective.

<u>Thm.</u> (Riemann) Suppose $\sum a_n$ converges conditionally. Then there is a rearrangement $\sum \tilde{a}_n$ s.t. $\sum \tilde{a}_n = \alpha$ where $\alpha \in [-\infty, \infty]$ or where $\sum \tilde{a}_n$ oscillates.



Choose $\alpha_n \searrow \alpha$ $\qquad \alpha \geq \beta$

$\beta_n \nearrow \beta$

$\beta_{n-1} < \alpha_n$

<u>Thm.</u> If $\sum a_n = a$ converges absolutely, then any rearrangement converges to the same value.