disc. 1

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D, C, f

$f: \mathbb{R}^+ \to \mathbb{R}$
$f(x) = x^2$

Some material from §7.2

$(f \circ g)(x) = f(g(x))$

Definition of inverse

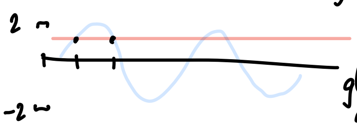
A function $f: D \to C$ has an inverse if there exists a fn. f^{-1} s.t. $\underbrace{f \circ f^{-1} = id}_{f(f^{-1}(y)) = y}$, $\underbrace{f^{-1} \circ f = id}_{f^{-1}(f(x)) = x}$

$f^{-1}: C \to D$

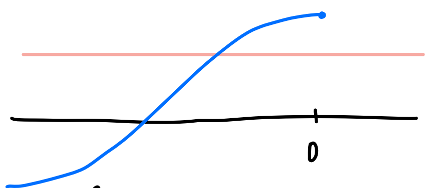
Injective & surjective functions $f: D \to C$

We say that a function is injective or one-to-one if it satisfies:

$f(x) = f(y) \Rightarrow x = y.$



$g(x) = 2\cos x$
$g: \mathbb{R} \to \mathbb{R}$



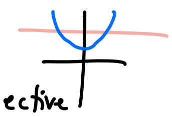
$\{x : x \le 0\}$

We say a function is surjective or onto if for every $y \in C$, there exists $x \in D$ s.t. $f(x) = y$. The range of f is $f(D) = \{f(x) : x \in D\}$.

Thus f is onto if and only if the range is equal to the codomain.

Ex. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 1$.

not injective, not surjective



$f: \mathbb{R} \to [1, \infty)$, $f(x) = x^2 + 1$ not injective, surjective

$f: \mathbb{R}^+ \to \mathbb{R}$, $f(x) = x^2 + 1$ injective, not surjective

$\{x \in \mathbb{R} : x > 0\}$

$f: \mathbb{R}^+ \to [1, \infty)$, $f(x) = x^2 + 1$ injective & surjective

Defining an inverse

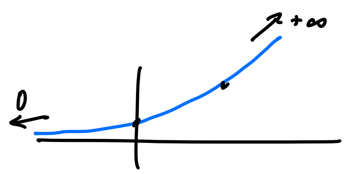
A function is invertible iff. it is both injective and surjective.

bijective

To construct the inverse $f^{-1}: C \to D$,

For $y \in C$: $f^{-1}(y) :=$ the unique x s.t. $f(x) = y$.

$x < y \Rightarrow f(x) < f(y)$



Note: $f: C \to f(C)$ is invertible iff. f is injective.

Example: exponential $e^x: \mathbb{R} \to \mathbb{R}^+$ has an inverse $\ln: \mathbb{R}^+ \to \mathbb{R}$

$e^0 = 1$

$e^x \to \infty$ as $x \to \infty$

Inverse function thm. Suppose f is invertible and differentiable, y is in the domain, and $f'(f^{-1}(y)) \neq 0$.

Then $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$

Ex. $f: \mathbb{R} \to \mathbb{R}^+$ via $f(x) = e^x$, $f^{-1}: \mathbb{R}^+ \to \mathbb{R}$ $f^{-1}(x) = \ln x$

$$\frac{d}{dy} \ln y = (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f(f^{-1}(y))} = \frac{1}{y}$$

$f'(x) = e^x$ $\{x \in \mathbb{R} : x > 0\}$

Ex. 1

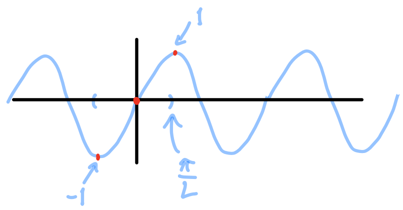
a. §7.2#3: What is the largest interval containing zero on which $f(x) = \sin x$ $: \mathbb{R} \to \mathbb{R}$ is one-to-one?

b. §7.2#2: Is $f(x) = x^2 + 2$ one-to-one? If not, describe a domain on which it is one-to-one.

$f: \mathbb{R} \to \mathbb{R}$

Does f have an inverse? On what domain?

a.



$[-\frac{\pi}{2}, \frac{\pi}{2}]$

Ex. 2 Let $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be defined by $f(x) = 2 \tan x$. What is f^{-1}?

$y = 2 \tan x$

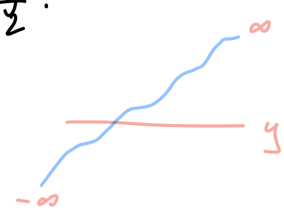
$\Rightarrow \tan x = \frac{y}{2}$

$\Rightarrow x = \arctan \frac{y}{2}$ $f^{-1}(y) = \arctan \frac{y}{2}$.

Ex. 3 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 2\sin x$.

(a) Show that f has an inverse.

(b) Find $(f^{-1})'(0)$.

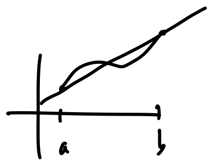


To see it's one-to-one: note that $f'(x) = 3 + 2\cos x$

$\geq 3 + 2(-1) = 1$

Let $x < y$: by the MVT,

there exists $c \in (x, y)$ such that



$$\frac{f(y) - f(x)}{y - x} = f'(c) \geq 1$$

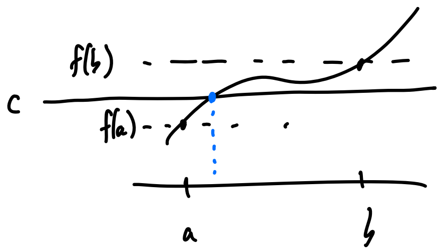
$$\Rightarrow \quad f(y) - f(x) \geq y - x > 0$$

$$\Rightarrow \quad f(x) < f(y)$$

To see it is surjective:

note that $f \to \infty$ as $x \to +\infty$

$f \to -\infty$ as $x \to -\infty$



f cont.

disc. 1

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$$f: \underset{\substack{\uparrow \\ \text{domain}}}{D} \to \underset{\substack{\nwarrow \\ \text{codomain}}}{C} \qquad \text{e.g. } f: \mathbb{R} \to \mathbb{R} \quad f(x) = x^2$$

Some material from §7.2

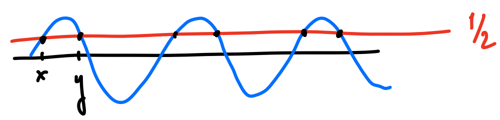
$$(f \circ g)(x) = f(g(x))$$

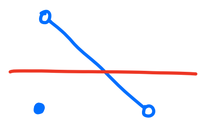
Definition of inverse

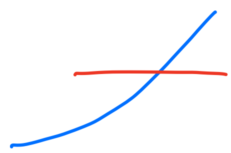
We say $f: D \to C$ is invertible if there exists a function $f^{-1}: C \to D$ such that
$$(f \circ f^{-1})(y) = y, \quad (f^{-1} \circ f)(x) = x$$

Injective & surjective functions $f: D \to C$

We say f is injective or one-to-one if $f(x) = f(y) \Rightarrow x = y$.





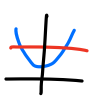


A function f is surjective or onto if for every $y \in C$, there exists x such that $f(x) = y$.

The range of f is the set $f(D) = \{f(x) : x \in D\}$.

A function is surjective iff. the range is equal to the codomain.

Ex. $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2 + 1$



not injective
not surjective

$f: \mathbb{R} \to [1, \infty)$ $f(x) = x^2 + 1$

not injective
surjective

$f: \mathbb{R}^{\geq 0} \to [1, \infty)$ $f(x) = x^2 + 1$

$\{x \in \mathbb{R} : x \geq 0\}$

injective
surjective

$f: \mathbb{R}^{\geq 0} \to \mathbb{R}$ $f(x) = x^2 + 1$

injective
not surjective

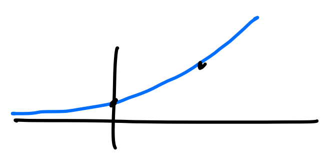
Defining an inverse

A function is invertible iff. it is injective and surjective.

Let $y \in C$. Then $f^{-1}(y) :=$ the unique x s.t. $f(x) = y$.

$$f(f^{-1}(y)) = f(x) = y$$
$$f^{-1}(f(x)) = x$$

Note: $f: D \to f(D)$ is invertible iff. it is one-to-one.



Example: exponential $e^x: \mathbb{R} \to \mathbb{R}^+$ has inverse $\ln x: \mathbb{R}^+ \to \mathbb{R}$

Inverse function thm. Suppose f is invertible and differentiable, $y \in C$, $f'(f^{-1}(y)) \neq 0$.

Then $\quad (f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$.

Ex. Let $f: \mathbb{R} \to \mathbb{R}^+$ be e^x, $f^{-1}(y) = \ln y$.
Then
$$\frac{d}{dy} \ln y = (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f(f^{-1}(y))} = \frac{1}{y}.$$
$\uparrow$
$f' = f$

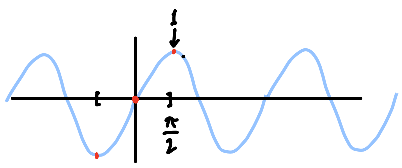
Ex. 1

$:\mathbb{R} \to \mathbb{R}$

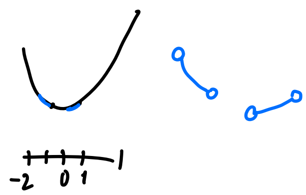
a. §7.2#3: What is the largest interval containing zero on which $f(x) = \sin x$ is one-to-one?

b. §7.2#2: Is $f(x) = x^2 + 2$ one-to-one? If not, describe a domain on which it is one-to-one.
$=$ $f: \mathbb{R} \to \mathbb{R}$
Does f have an inverse? On what domain?

a.



$[-\frac{\pi}{2}, \frac{\pi}{2}]$

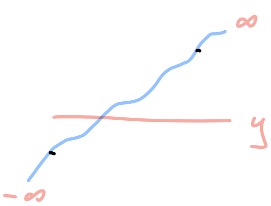


$(0, 1) \cup (-2, -1)$

b. Not 1-1. E.g. $\{x \in \mathbb{R} : x \geq 0\}$ $\quad \{x \in \mathbb{R} : x > 0\}$
$\{x \in \mathbb{R}, x \leq 0\}$ $\quad \{x \in \mathbb{R} : x > 5\}$

Ex. 2 Let $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be defined by $f(x) = 2 \tan x$. What is f^{-1}?
$$y = 2\tan x \Rightarrow \tan x = \frac{y}{2} \Rightarrow f^{-1}(y) = x = \arctan \frac{y}{2}.$$

Ex. 3 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 2\sin x$.
(a) Show that f has an inverse.



(b) Find $(f^{-1})'(0)$. $\quad \downarrow$

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(0)} = \frac{1}{5}$$

$f^{-1}: f(\mathbb{R}) \to \mathbb{R}$

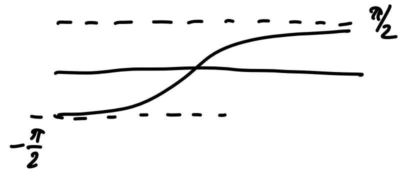
one-to-one: $\quad f'(x) = 3 + 2\cos x \geq 3 + 2(-1) = 1 > 0$

onto: $\quad$ Note that $f(x) \to \infty$ as $x \to +\infty$
$f(x) \to -\infty$ as $x \to -\infty$
and use the intermediate value theorem.

Ex. 4 $g: \mathbb{R} \to \mathbb{R}$ via $g(x) = \arctan x$

Range is $g(\mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2})$ one-to-one: $g'(x) = \dfrac{1}{1+x^2} > 0$

Not onto.



But there is an inverse $g^{-1} : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, namely tan.

disc. 2 (site: bertrandstone.com/ta)

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

e^x, $\ln x$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$G'(x) = \frac{d}{dx}\int_1^x \frac{dt}{t} = \frac{1}{x}$$

Another interesting approach to the logarithm (from Rogawski §7.3 Ex. 116)

Define a function G for $x > 0$:

$$G(x) = \int_1^x \frac{1}{t}\,dt$$

$$\begin{cases} \ln(xy) = \ln x + \ln y \\ \frac{d}{dx}\ln x = \frac{1}{x} \\ \ln x^a = a\ln x \quad ? \end{cases}$$

116. Defining $\ln x$ as an Integral This exercise proceeds as if we didn't know that $G(x) = \ln x$ and shows directly that G has all the basic properties of the logarithm. Prove the following statements.

(a) $\displaystyle\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{t}\,dt$ for all $a, b > 0$. Hint: Use the substitution $u = t/a$.

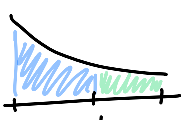
$u = t/a$
$du = dt/a$

$$= \int_1^b \frac{1}{au}\,a\,du = \int_1^b \frac{du}{u}$$

(b) $G(ab) = G(a) + G(b)$. Hint: Break up the integral from 1 to ab into two integrals and use (a).

$$\boxed{\int_b^a = -\int_a^b}$$

$$G(ab) = \int_1^{ab}\frac{dt}{t} = \int_1^a \frac{dt}{t} + \underbrace{\int_a^{ab}\frac{dt}{t}}_{\overset{(a)}{=}\int_1^b \frac{dt}{t} = G(b)} = G(a) + G(b)$$



(c) $\underbrace{G(1) = 0}$ and $G(a^{-1}) = -G(a)$ for $a > 0$.

$$\int_1^1 \frac{dt}{t} = 0 \qquad 0 = G(1) = G(aa^{-1}) = G(a) + G(a^{-1})$$

$$\Rightarrow G(a^{-1}) = -G(a).$$

(d) $G(a^n) = nG(a)$ for all $a > 0$ and integers n. (clear for $n = 0$)

$n > 0$: $G(a^n) = G(aa^{n-1}) = G(a) + G(a^{n-1}) = \cdots = G(a) + \cdots + G(a) = nG(a)$

$\underbrace{\hphantom{xxxxxxxxxx}}_{n \text{ times}}$

$n < 0$: $\quad G(a^n) = G((a^{-1})^{-n}) \overset{\checkmark}{=} -n\,G(a^{-1}) \overset{(c)}{=} -n(-G(a))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = n\,G(a)$

(e) $G(a^{1/n}) = \frac{1}{n} G(a)$ for all $a > 0$ and integers $n \neq 0$.

$$G(a) = G((a^{1/n})^n) \overset{(d)}{=} n\,G(a^{1/n}) \quad \Rightarrow \quad G(a^{1/n}) = \frac{1}{n} G(a).$$

(f) $G(a^r) = r\,G(a)$ for all $a > 0$ and rational numbers r.

$r = \frac{p}{q}$, p, q integers, $q \neq 0$

$$G(a^r) = G(a^{p/q}) = G((a^{1/q})^p) \overset{(d)}{=} p\,G(a^{1/q}) \overset{(e)}{=} \frac{p}{q} G(a) = r\,G(a).$$

(g) G is strictly increasing. $\quad (x < y \Rightarrow G(x) < G(y))$

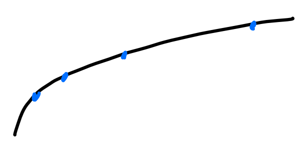
$G'(x) = \frac{d}{dx} \int_1^x \frac{1}{t}\, dt = \frac{1}{x} > 0.$ So G has an inverse, defined on its range.

(h) There exists a number a such that $G(a) > 1$. Hint: Show that $G(2) > 0$ and take $a = 2^m$ for $m > 1/G(2)$.

$G(2) = \int_1^2 \frac{dt}{t} > 0$

$$G(a) = G(2^m) = m\,G(2) > 1$$

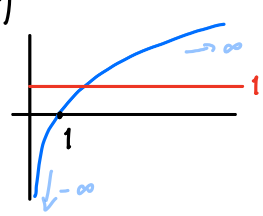
(i) $\lim_{x \to \infty} G(x) = \infty$ and $\lim_{x \to 0^+} G(x) = -\infty$. (Note that G is increasing.)



$$\lim_{m \to \infty} G(2^m) = \lim_{m \to \infty} m\,G(2) = \infty$$

$$\lim_{m \to \infty} G(2^{-m}) = \lim_{m \to \infty} (-m\,G(2)) = -\infty$$

(j) There exists a unique number E such that $G(E) = 1$.



By (i) and IVT, there exists an E

E is unique because G is strictly increasing

(k) $G(E^r) = r$ for every rational number r.

$G(E^r) = rG(E) = r$

Then, we can define e^x to be the inverse of $\ln x := G(x)$.

- $G(xy) = G(x) + G(y)$
- $G(x^r) = rG(x)$ (r rational)
- $G'(x) = \frac{1}{x}$
- The range of G is $\mathbb{R}$
- G is strictly increasing

117

G has an inverse $G^{-1} : \mathbb{R} \to \underbrace{\mathbb{R}^+}_{\{x \in \mathbb{R} : x > 0\}}$

$\ln x = G(x)$

$e^x = G^{-1}(x)$

$e^{x+y} = e^x e^y$

$f : D \to C$ $\quad (f \circ f^{-1})(y) = y$

$f^{-1} : C \to D$ $\quad (f^{-1} \circ f)(x) = x$

$(f^{-1})^{-1} = f$

$f(x) = 3x + 2\sin x$ $\quad f(0) = 3 \cdot 0 + 2\sin 0 = 0 \Rightarrow f^{-1}(0) = 0$

$f'(x) = 3 + 2\cos x$

We showed f^{-1} exists $\quad (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$ $\quad \underbrace{f^{-1}(f(0))}_{=0} = f^{-1}(0)$

(b). $(f^{-1})'(0) = \frac{1}{3 + 2\cos \underbrace{f^{-1}(0)}_{=0}} = \frac{1}{5}.$

disc. 2 (PDF to site: bertrandstone.com/ta)

$$G'(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

Another interesting approach to the logarithm (from Rogawski §7.3 Ex. 116)

Define a function G for $x > 0$:

$$G(x) = \int_1^x \frac{1}{t}\,dt$$

$G(xy) = G(x) + G(y)$

$G'(x) = \frac{1}{x}$

$G(x^r) = r\,G(x)$, r rational

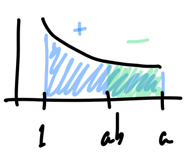
<u>116</u>. <u>Defining $\ln x$ as an Integral</u> This exercise proceeds as if we didn't know that $G(x) = \ln x$ and shows directly that G has all the basic properties of the logarithm. Prove the following statements.

(a) $\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{t}\,dt$ for all $a, b > 0$. Hint: Use the substitution $u = t/a$.

$$\int_a^{ab} \frac{1}{t}\,dt \overset{u = t/a}{\underset{du = \frac{dt}{a}}{=}} \int_1^b \frac{1}{\cancel{a}u}\,\cancel{a}\,du = \int_1^b \frac{du}{u}$$

(b) $G(ab) = G(a) + G(b)$. Hint: Break up the integral from 1 to ab into two integrals and use (a).

$$\int_b^a = -\int_a^b$$



$$G(ab) = \int_1^{ab} \frac{dt}{t} = \underbrace{\int_1^a \frac{dt}{t}}_{= G(a)} + \underbrace{\int_a^{ab} \frac{dt}{t}}_{\overset{(a)}{=} \int_1^b \frac{dt}{t}} = G(a) + G(b)$$

(c) $\underbrace{G(1) = 0}_{\int_1^1 \cdot = 0}$ and $G(a^{-1}) = -G(a)$ for $a > 0$.

$$0 = G(1) = G(aa^{-1}) \overset{(b)}{=} G(a) + G(a^{-1})$$

$$\Rightarrow G(a^{-1}) = -G(a).$$

(d) $G(a^n) = nG(a)$ for all $a > 0$ and integers n. (clear for $n = 0$)

$n > 0$: $a^n = aa^{n-1} \Rightarrow G(a^n) = G(aa^{n-1}) = G(a) + G(a^{n-1}) = \cdots = G(a) + \cdots + G(a)$

$$n < 0: \quad G(a^n) = G((a^{-1})^{-n}) = -n\,G(a^{-1}) \overset{(c)}{=} n\,G(a) \qquad \overset{n \text{ times}}{=} n\,G(a)$$

(e) $G(a^{1/n}) = \frac{1}{n} G(a)$ for all $a > 0$ and integers $n \neq 0$.

$$G(a) = G((a^{1/n})^n) \overset{(d)}{=} n\,G(a^{1/n}) \implies G(a^{1/n}) = \frac{1}{n} G(a)$$

(f) $G(a^r) = r\,G(a)$ for all $a > 0$ and rational numbers r.

$r = \frac{p}{q}$, p, q are integers, $q \neq 0$

$$G(a^r) = G(a^{p/q}) = G((a^p)^{1/q}) \overset{(e)}{=} \frac{1}{q} G(a^p) = \frac{p}{q} G(a) = r\,G(a).$$

(g) G is strictly increasing. $\quad (x < y \Rightarrow G(x) < G(y))$

$G'(x) = \frac{1}{x} > 0$ Hence, G has an inverse, defined on its range.

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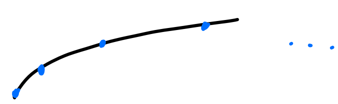
Note $G(2) = \int_1^2 \frac{dt}{t} > 0$. So for $a = 2^m$: $G(a) = G(2^m) = m\,G(2) > 1$ if $m > 1/G(2)$.

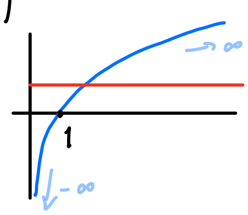
(i) $\lim_{x \to \infty} G(x) = \infty$ and $\lim_{x \to 0^+} G(x) = -\infty$. (Note that G is increasing.)

$$\lim_{m \to \infty} G(2^m) = \lim_{m \to \infty} m\,G(2) = \infty$$

$$\lim_{m \to \infty} G(2^{-m}) = \lim_{m \to \infty} -m\,G(2) = -\infty$$



(j) There exists a unique number E such that $G(E) = 1$. More generally, range of G is $\mathbb{R}$.



E exists by IVT

E is unique since G is strictly increasing.

(k) $G(E^r) = r$ for every rational number r.

$$G(E^r) = r\,G(E) = r$$

Range of G is $\mathbb{R}$
G has an inverse
$: \mathbb{R} \to \mathbb{R}^+$

Then, we can define e^x to be the inverse of $\ln x := G(x)$.

$$e^x := G^{-1}(x)$$

$$e^{x+y} = e^x e^y$$

$$\frac{d}{dx} e^x = \frac{1}{G'(G^{-1}(x))} = G^{-1}(x) = e^x.$$

Suppose f' is Riemann integrable on $[a, b]$.

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

If f is integrable and $F' = f$, then

$$\int_a^b \underset{F'(x)}{f(x)}\,dx = F(b) - F(a)$$

Suppose f is continuous. We have

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Ex. $$\int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}.$$

$$\frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3$$

$$f(x)\Big|_a^b = f(b) - f(a)$$

Ex. $$\frac{d}{dx}\int_1^x \cos t\,dt = \cos x$$

Ex. $$\frac{d}{dx}\underbrace{\int_1^{x^2} \cos t\,dt}_{\left(\int_1^{\cdot} \cos t\,dt\right) \circ x^2} = (\cos x^2)\,2x$$

disc. 3

L'Hôpital's rule

Suppose f, g are differentiable on (a, b), where $a < b$, and $c \in (a, b)$, $g'(x) \neq 0$ for x near c (except perhaps at $x = c$).

If (i) $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$

or

(ii) $\lim_{x \to c} g(x) = \infty$

then if $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists, we have

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

(The rule also works for one-sided limits, or limits at ∞).

$\frac{\infty}{\infty}$ $\frac{0}{0}$

rewrite as quotient

$0 \cdot \infty$

take log

0^0 1^∞

Ex. 1 $\lim_{x \to \infty} \frac{x^2}{e^x} \overset{L'H}{=} \lim_{x \to \infty} \frac{2x}{e^x} \overset{L'H}{=} \lim_{x \to \infty} \frac{2}{e^x} = 0$

Ex. 2 $\lim_{x \to 0^+} x^{x^2}$ (recall $\lim_{x \to 0^+} x^x = 1$)

Consider $\lim_{x \to 0^+} \ln\left(x^{x^2}\right) = \lim_{x \to 0^+} x^2 \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x^2} \overset{L'H}{=} \lim_{x \to 0^+} \frac{1/x}{-\frac{2}{x^3}} = -\frac{1}{2} \lim_{x \to 0^+} x^2 = 0$

Since $x^{x^2} = e^{x^2 \ln x}$, we have:

$$\lim_{x \to 0^+} x^{x^2} = \lim_{x \to 0^+} e^{x^2 \ln x} = e^{\lim_{x \to 0^+} x^2 \ln x} = e^0 = 1.$$

Note: Möbius claimed if $f(x) \to 0$, $g(x) \to 0$ as $x \to 0^+$, then $f(x)^{g(x)} \to 1$ as $x \to 0^+$

Ex. Show that this is false. $f(x) = e^{-1/x}$, $g(x) = x$

$f(x)^{g(x)} = \left(e^{-1/x}\right)^x = e^{-1} \not\to 1$

Ex. 3 $\lim_{x \to \infty} \frac{\ln x + \sin x}{x} = 0 \neq \lim_{x \to \infty} \frac{1/x + \cos x}{1}$ does not exist

$\lim_{x \to \infty} \frac{\sin x}{x}$ $\quad \frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$ so $\frac{\sin x}{x} \to 0$ by squeeze thm.

$\to 0$ $\quad \to 0$

$\lim_{x \to \infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x \to \infty} \frac{1/x}{1} = 0$

Ex. 4 $\quad \lim_{x\to 0^+} \dfrac{e^{-1/x}}{x} \qquad \lim_{x\to 0^+} \dfrac{e^{-1/x}(x^{-2})}{1} = \lim_{x\to 0^+} \dfrac{e^{-1/x}}{x^2}$

$$\downarrow$$

$$\lim_{x\to 0^+} \frac{1/x}{e^{1/x}} \overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{(-1/x^2)}{e^{1/x}(-1/x^2)} = \lim_{x\to 0^+} \frac{1}{e^{1/x}} = 0$$

Application: an interesting function

$$f(x) = \begin{cases} e^{-1/|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f'(0)$? $\qquad \lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \dfrac{e^{-1/|x|}}{x} \quad \begin{cases} \text{right lim.} \quad \lim_{x\to 0^+} \dfrac{e^{-1/x}}{x} = 0 \quad \text{by above} \\ \text{left lim.} \quad -\lim_{x\to 0^-} \dfrac{e^{-1/(-x)}}{(-x)} = 0 \end{cases}$

so that $f'(0) = 0$.

In fact, f is infinitely differentiable and $f^{(n)}(0) = 0$.

Ex. 5 $\quad \lim_{x\to\infty} \left(1 + \dfrac{r}{x}\right)^x$

Consider $\lim_{x\to\infty} \ln\left(\left(1+\dfrac{r}{x}\right)^x\right) = \lim_{x\to\infty} x \ln\left(1 + \dfrac{r}{x}\right) = \lim_{x\to\infty} \dfrac{\ln(1 + r/x)}{1/x}$

$$\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+r/x}\left(-\frac{r}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{r}{1 + r/x} = r$$

so $\lim_{x\to\infty} \left(1+\dfrac{r}{x}\right)^x = e^{\lim_{x\to\infty} \ln\left(\left(1+\frac{r}{x}\right)^x\right)} = e^r$

Ex. 6 $\quad \lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} \qquad \lim_{x\to\infty} \dfrac{1}{\cancel{\frac{1}{2}}(x^2+1)^{-1/2}\cdot \cancel{2}x} = \lim_{x\to\infty} \dfrac{(x^2+1)^{1/2}}{x}$

$\sqrt{x^2+1} = \sqrt{x^2\left(1+\frac{1}{x^2}\right)} = x\sqrt{1+\frac{1}{x^2}} \qquad \lim_{x\to\infty} \dfrac{\cancel{\frac{1}{2}}(x^2+1)^{-1/2}\cancel{2}x}{1} = \lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\sqrt{1+\frac{1}{x^2}}} = 1.$$

$$\lim_{x\to\infty} \frac{e^x}{e^x + e^{2x}} \cdot \frac{e^{-x}}{e^{-x}}$$

$$\frac{1}{1 + e^x}$$

Ex. 7 $\quad \lim_{x\to\infty} \dfrac{x^7 + 6x^6 + 20x^3 - x^2 + x - 7}{x^8 + 4x^4 - 3} \cdot \dfrac{x^{-8}}{x^{-8}}$

disc. 3

L'Hôpital's rule

Suppose f, g are differentiable on (a, b), where $a < b$, and $c \in (a, b)$, $g'(x) \neq 0$ for x near c (except perhaps at $x = c$).

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(The rule also works for one-sided limits, or limits at ∞).

$\frac{\infty}{\infty}$ $\quad$ $\frac{0}{0}$

rewrite as quotient

$0 \cdot \infty$

take log

0^0 $\quad$ 1^∞

Ex. 1 $\displaystyle \lim_{x \to \infty} \frac{x^2}{e^x} \overset{L'H}{=} \lim_{x \to \infty} \frac{2x}{e^x} \overset{L'H}{=} \lim_{x \to \infty} \frac{2}{e^x} = 0$

Ex. 2 $\displaystyle \lim_{x \to 0^+} x^{x^2}$ $\quad$ (also $\lim_{x \to 0^+} x^x = 1$)

Consider $\displaystyle \lim_{x \to 0^+} \ln(x^{x^2}) = \lim_{x \to 0^+} x^2 \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x^2} \overset{L'H}{=} \lim_{x \to 0^+} \frac{x^{-1}}{-2x^{-3}} = -\frac{1}{2} \lim_{x \to 0^+} x^2 = 0$

$$\lim_{x \to 0^+} x^{x^2} = \lim_{x \to 0^+} e^{x^2 \ln x} = e^{\lim_{x \to 0^+} x^2 \ln x} = e^0 = 1$$

Note: Möbius claimed if $f(x) \to 0$, $g(x) \to 0$ as $x \to 0^+$, then $f(x)^{g(x)} \to 1$ as $x \to 0^+$

Ex. Show that this is false. $\quad f(x) = e^{-1/x}$, $g(x) = x$

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$$\lim_{x \to \infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x \to \infty} \frac{1/x}{1} = 0$$

$\displaystyle \lim_{x \to \infty} \frac{\sin x}{x} = 0$ by squeeze thm. $\quad \underbrace{\frac{-1}{x}}_{\to 0} \leq \frac{\sin x}{x} \leq \frac{1}{x} \to 0$

Ex. 4 $\displaystyle\lim_{x\to 0^+} \frac{e^{-1/x}}{x}$ $\qquad \displaystyle\lim_{x\to 0^+} \frac{e^{-1/x}(x^{-2})}{1} = \lim_{x\to 0^+} \frac{e^{-1/x}}{x^2}$

$$\| $$

$$\lim_{x\to 0^+} \frac{1/x}{e^{1/x}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{(-x^{-2})}{e^{1/x}(-x^{-2})} = \lim_{x\to 0^+} \frac{1}{e^{1/x}} = 0$$

Application: an interesting function

$$f(x) = \begin{cases} e^{-1/|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f'(0)$?

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x} \qquad \begin{cases} \displaystyle\lim_{x\to 0^+} \frac{e^{-1/x}}{x} = 0 \\ \displaystyle\lim_{x\to 0^-} \frac{e^{1/x}}{x} = \lim_{x\to 0^+} \frac{e^{1/(-x)}}{(-x)} = -\lim_{x\to 0^+} \frac{e^{-1/x}}{x} = 0 \end{cases}$$

$$\Rightarrow f'(0) = 0$$

In fact, we can show that f is infinitely differentiable and $f^{(n)}(0) = 0$

Ex. 5 $\displaystyle\lim_{x\to\infty} \left(1 + \frac{r}{x}\right)^x$

Consider $\displaystyle\lim_{x\to\infty} x \ln\left(1 + \frac{r}{x}\right) = \lim_{x\to\infty} \frac{\ln(1 + r/x)}{1/x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{r}{1 + r/x}\cancel{(-x^{-2})}}{\cancel{(-x^{-2})}} = r$

So,

$$\lim_{x\to\infty} \left(1 + \frac{r}{x}\right)^x = e^{\lim_{x\to\infty} x \ln\left(1 + \frac{r}{x}\right)} = e^r$$

Ex. 6 $\displaystyle\lim_{x\to\infty} \frac{e^x}{e^x + e^{2x}}$ $\qquad \displaystyle\lim_{x\to\infty} \frac{e^x}{e^x + 2e^{2x}}$ $\qquad \displaystyle\lim_{x\to\infty} \frac{e^x}{e^x + 4e^{2x}}$

$$\frac{e^x}{e^x + e^{2x}} \cdot \frac{e^{-x}}{e^{-x}} = \frac{1}{1 + e^x} \to 0$$

Ex. 7 $\displaystyle\lim_{x\to\infty} \frac{x^7 + 6x^6 + 20x^3 - x^2 + x - 7}{x^8 + 4x^4 - 3} \cdot \frac{x^{-8}}{x^{-8}} = 0$

disc. 4

Integration by parts $u(b)v(b) - u(a)v(a) = \int_a^b (uv)'(t)\,dt = \int_a^b u'(t)v(t)\,dt + \int_a^b v'(t)u(t)\,dt$

$$\Rightarrow \int_a^b u(t)v'(t)\,dt = u(t)v(t)\Big|_a^b - \int_a^b u'(t)v(t)\,dt$$

For indefinite integrals, $\int uv' = uv - \int u'v$

Ex. 1 $\int \ln x\,dx = \int \underbrace{\ln x}_{u} \cdot \underbrace{1}_{v'}\,dx = x\ln x - \int \frac{1}{x} x\,dx = x\ln x - x + c$

$u' = \frac{1}{x}$, $v = x$

Ex. 2 (gamma function) $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \quad (x > 0)$

Suppose $x > 0$. Integrating by parts

$$\Gamma(x+1) = \int_0^\infty \underbrace{t^x}_{u}\underbrace{e^{-t}}_{v'}\,dt$$

$u' = xt^{x-1}$, $v = -e^{-t}$

$$= -t^x e^{-t}\Big|_0^\infty - \int_0^\infty x t^{x-1}(-e^{-t})\,dt$$

$$= x\int_0^\infty t^{x-1}e^{-t}\,dt = x\Gamma(x)$$ (recursion formula for the gamma function)

$(n \geq 1)$ So $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)(n-2)\cdots 1 = n!$

Can define $x! = \Gamma(x+1)$ for x not necessarily an integer $(x \geq 0)$

E.g. $\left(\frac{5}{2}\right)! = \Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\underbrace{\Gamma\left(\frac{1}{2}\right)}_{=\sqrt{\pi}}$ (using a change of vars. and $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$)

$= \frac{15}{8}\sqrt{\pi}$.

35. $\int e^{\sqrt{x}}\,dx \overset{u=\sqrt{x}}{\underset{du=\frac{dx}{2\sqrt{x}}}{=}} \int e^u 2u\,du = 2\int \underbrace{u}_{f}\underbrace{e^u}_{g'}\,du \quad f' = 1,\ g = e^u$

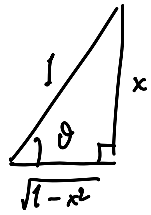
$$= 2\left(ue^u - \int 1\cdot e^u\,du\right)$$

$$= 2(ue^u - e^u) + c$$

$$= 2(u-1)e^u + c$$

$$= 2(\sqrt{x}-1)e^{\sqrt{x}} + c$$

81. $\int \arcsin^2 x \, dx = \int \underbrace{\arcsin^2 x}_{u} \cdot \underbrace{1}_{v'} \, dx$



$\arcsin x = \theta$

$\cos \theta = ?$

$\cos \theta = \sqrt{1-x^2}$

$a^2 + x^2 = 1^2$

$a^2 = 1 - x^2$

$$u' = \frac{2 \arcsin x}{\sqrt{1-x^2}}, \quad v = x$$

$$= x \arcsin^2 x - 2 \underbrace{\int \frac{x \arcsin x}{\sqrt{1-x^2}} \, dx}$$

$u = \arcsin x$

$du = \frac{1}{\sqrt{1-x^2}} \, dx$

$\int \underbrace{u}_{f} \underbrace{\sin u}_{g'} \, du \qquad f' = 1, \; g = -\cos u$

$= -u \cos u + \int \cos u \, du$

$= -u \cos u + \sin u + \tilde{C}$

$$= x \arcsin^2 x - 2(-\arcsin x \cos \arcsin x + \sin \arcsin x + \tilde{C})$$

$$= x \arcsin^2 x + 2\sqrt{1-x^2} \arcsin x - 2x + C$$

<u>Trig. integrals</u>

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx \qquad (n \geq 2)$$

So, we can compute $\int \cos^n x \, dx$, $\int \sin^n x \, dx$ for any $n \geq 0$.

$9 \to 7 \to 5 \to 3 \to 1$

$8 \to 6 \to 4 \to 2 \to 0$

$$\int \sin^m x \cos^n x \, dx$$

m odd: Split off one $\sin x$, rewrite $\sin^{m-1} x$ in terms of $\cos x$ & set $u = \cos x$

n odd: Same thing but split off $\cos x$ instead

m, n even: Use $\cos^2 + \sin^2 = 1$ to write in terms of powers of sin or cos.

$$\int \cos^n x \underbrace{\sin^{m-1} x}_{(\sin^2 x)^{\frac{m-1}{2}}} \sin x \, dx = \int \cos^n x (1 - \cos^2 x)^{\frac{m-1}{2}} \sin x \, dx \overset{u = \cos x}{\underset{du = -\sin x \, dx}{=}} -\int u^n (1-u^2)^{\frac{m-1}{2}} \, du$$

$$\int \cos^n x \sin^m x \, dx = \int \cos^n x (1 - \cos^2 x)^{m/2} \, dx$$

disc. 4

Integration by parts $u(b)v(b) - u(a)v(a) = \int_a^b (uv)'(t)\,dt = \int_a^b u'(t)v(t) + u(t)v'(t)\,dt$

$$\Rightarrow \int_a^b u(t)v'(t)\,dt = u(t)v(t)\Big|_a^b - \int_a^b u'(t)v(t)\,dt$$

For indefinite integrals: $\int uv' = uv - \int u'v$

Ex. 1 $\int \ln x\,dx = \int \underbrace{\ln x}_{u} \cdot \underbrace{1}_{v'}\,dx = x\ln x - \int \frac{1}{x}\cdot x\,dx = x\ln x - x + C$

$u' = \frac{1}{x} \quad v = x$

Ex. 2 (gamma function) $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \quad (x > 0)$

We will apply integration by parts to

$$\Gamma(x+1) = \int_0^\infty \underbrace{t^x}_{u}\underbrace{e^{-t}}_{v'}\,dt$$

$u' = xt^{x-1}, \quad v = -e^{-t}$

$$= \cancel{-t^x e^{-t}\Big|_0^\infty} - \int_0^\infty xt^{x-1}(-e^{-t})\,dt$$

0

$$= x\int_0^\infty t^{x-1}e^{-t}\,dt$$

$$= x\Gamma(x)$$

$$\Gamma(1/2) = \int_0^\infty t^{-1/2}e^{-t}\,dt \overset{t=u^2}{\underset{dt = 2u\,du}{=}} \int_0^\infty \frac{1}{\cancel{u}}e^{-u^2}\,2\cancel{u}\,du$$

$$= 2\int_0^\infty e^{-u^2}\,du$$

$$= \int_{-\infty}^\infty e^{-u^2}\,du$$

$$= \sqrt{\pi}$$

Suppose n is a positive integer. Then

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2)$$

$$= \cdots = n(n-1)(n-2)\cdots 1 = n!$$

So we can define a factorial for non-integer x: $x! = \Gamma(x+1)$

E.g. $\left(\frac{5}{2}\right)! = \Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$

35. $\int e^{\sqrt{x}}\,dx \overset{u=\sqrt{x}}{\underset{du = \frac{dx}{2\sqrt{x}}}{=}} \int e^u \cdot 2u\,du = 2\int \underbrace{u}_{f}\underbrace{e^u}_{g'}\,du \qquad f' = 1, \; g = e^u$

$2u\,du = dx$

$$= 2\left(ue^u - \int 1\cdot e^u\,du\right)$$

$$= 2(ue^u - e^u) + C = 2(\sqrt{x} - 1)e^{\sqrt{x}} + C$$

81. $\int \arcsin^2 x \, dx$

$$\int \underbrace{\arcsin^2 x}_{u} \cdot \underbrace{1}_{v'} \, dx = x \arcsin^2 x - 2 \underbrace{\int \frac{x \arcsin x}{\sqrt{1-x^2}} \, dx}$$

$u' = \frac{2 \arcsin x}{\sqrt{1-x^2}}$ $\quad v = x$

$x = \sin u \leftrightarrow u = \arcsin x$

$du = \frac{dx}{\sqrt{1-x^2}}$

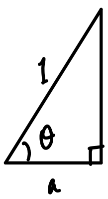
$$= \int \underbrace{u}_{f} \underbrace{\sin u}_{g'} \, du$$

$f' = 1, \; g = -\cos u$

$$= -u \cos u + \int \cos u \, du$$

$$= -u \cos u + \sin u + \tilde{c}$$

$\cos(\underbrace{\arcsin x}_{=:\theta}) = ?$



$\theta = \arcsin x$
$\leftrightarrow$
$\sin \theta = x$

$\cos \theta = a$
$= \sqrt{1-x^2}$

$a^2 + x^2 = 1^2$

$\Rightarrow a^2 = 1 - x^2$

$\Rightarrow a = \sqrt{1-x^2}$

$$= x \arcsin^2 x - 2\left(-\arcsin x \cos \arcsin x + \sin \arcsin x + \tilde{c}\right)$$

$$= x \arcsin^2 x + 2(\cos \arcsin x) \arcsin x - 2x + c \quad (\text{write } c = -2\tilde{c})$$

$$= x \arcsin^2 x + 2\sqrt{1-x^2} \arcsin x - 2x + c$$

disc. 5

$\int \frac{dx}{x^3 - 3x^2 + 4}$

$$x^3 - 3x^2 + 4 = (x+1)(x^2 + ax + b)$$
$$= x^3 + ax^2 + bx + x^2 + ax + b$$

$$a + b = 0 \qquad b = 4$$
$$\Rightarrow a = -4$$

$$(x+1)(x^2 - 4x + 4) = (x+1)(x-2)^2$$

Or alternatively

$$\begin{array}{r|l} & x^2 - 4x + 4 \\ \hline x+1 & x^3 - 3x^2 + 4 \\ & -(x^3 + x^2) \\ \hline & 0 - 4x^2 \\ & -(-4x^2 - 4x) \\ \hline & + 4x + 4 \\ & -(4x + 4) \\ \hline & 0 \end{array}$$

$$\frac{1}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

multiply by this

$$1 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$
$$= A(x^2 - 4x + 4) + B(x^2 - x - 2) + Cx + C$$
$$0 = \underbrace{(A + B)}_{=0}x^2 + (\underbrace{-4A - B + C}_{=0})x + (\underbrace{4A - 2B + C - 1}_{=0})$$

$$A = -B$$

$$\begin{cases} -4(-B) - B + C = 0 \\ 4(-B) - 2B + C - 1 = 0 \end{cases}$$

$$\begin{cases} 3B + C = 0 \\ -6B + C - 1 = 0 \end{cases} \Rightarrow C = -3B$$
$$-6B - 3B = 1$$
$$\Rightarrow B = -1/9, \quad A = 1/9, \quad C = 1/3$$

$$\int \frac{dx}{(x+1)(x-2)^2} = \int \frac{1/9}{x+1} - \frac{1/9}{x-2} + \frac{1/3}{(x-2)^2}\, dx$$

$$= \frac{1}{9}\ln|x+1| - \frac{1}{9}\ln|x-2| - \frac{1}{3}(x-2)^{-1} + c$$

$$\frac{d}{dx}\ln x = \frac{1}{x} \quad (x > 0)$$
$$\frac{d}{dx}\ln|x| = \frac{1}{x} \quad (x \neq 0)$$

$\int \frac{100x}{(x-3)(x^2+1)^2}\, dx$

$$\frac{A}{x-3} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2} = \frac{100x}{(x-3)(x^2+1)^2}$$

$$A(x^2+1)^2 + (Bx + C)\underbrace{(x^2+1)(x-3)}_{x^3 + x - 3x^2 - 3} + (Dx + E)(x-3) = 100x$$

$$Ax^4 + 2Ax^2 + \underline{A} + Bx^4 + Bx^2 - 3Bx^3 - 3Bx + Cx^3 + Cx - 3Cx^2 - \underline{3C}$$
$$+ Dx^2 + Ex - 3Dx - \underline{3E} - 100x = 0$$

$$\underbrace{(A + B)}_{=0}x^4 + \underbrace{(C - 3B)}_{=0}x^3 + \underbrace{(2A + B - 3C + D)}_{=0}x^2 + (\underbrace{-3B + C + E - 3D - 100}_{=0})x + \underbrace{(A - 3C - 3E)}_{=0} = 0$$

$A = -B \qquad C = 3B \qquad \ldots$ Solving this gives $B = -3$

$$\Rightarrow \quad A = 3,\ B = -3,\ C = -9,\ D = -30,\ E = 10$$

$$\int \frac{100x}{(x-3)(x^2+1)^2}\,dx = \int \frac{3}{x-3} + \frac{-3x-9}{x^2+1} + \frac{-30x+10}{(x^2+1)^2}\,dx$$

$$= 3\underbrace{\int \frac{dx}{x-3}}_{\ln|x-3|} - 3\underbrace{\int \frac{x\,dx}{x^2+1}}_{} - 9\underbrace{\int \frac{dx}{x^2+1}}_{\arctan x} - 30\underbrace{\int \frac{x\,dx}{(x^2+1)^2}}_{} + 10\underbrace{\int \frac{dx}{(x^2+1)^2}}_{x = \tan\theta}$$

$u = x^2+1$, $du = 2x\,dx$

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|x^2+1| + c$$

$u = x^2+1$, $du = 2x\,dx$

$$\frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2}(x^2+1)^{-1} + c$$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$\tan^2 + 1 = \sec^2$

$(\sin^2 + \cos^2 = 1)$

$$\int \frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \int \cos^2\theta\,d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta) + c$$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}}\right) + c$$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + c$$

so $$\int \frac{100x}{(x-3)(x^2+1)^2}\,dx = 3\ln|x-3| - \frac{3}{2}\ln|x^2+1| - 4\arctan x + \frac{15}{x^2+1} + \frac{5x}{x^2+1} + c$$

What are the possible orders for irreducible polynomials over $\mathbb{R}$?

Answer: only 1 or 2

- <u>Fundamental thm. of algebra</u>

  If P is a nonconstant polynomial with coefficients in $\mathbb{C}$, then P has a root in $\mathbb{C}$.

- If z is a root of a <u>real</u> polynomial P, then so is $\bar{z}$ $\quad \begin{pmatrix} z = a + bi \\ \bar{z} = a - bi \end{pmatrix}$

Let P be a polynomial with real coefficients, of order ≥ 3. $\quad z + \bar{z} = 2a$

If P has a real root a, factor out $(x-a)$ $\longrightarrow$ done. $\quad z\bar{z} = a^2 + b^2$

If P has no real roots, let z be a complex root: this exists by FTA.

Moreover, $\bar{z}$ is also a root. So factor out $(x-z)(x-\bar{z})$:

this is a real irreducible quadratic because $(x-z)(x-\bar{z}) = x^2 - (z+\bar{z})x + z\bar{z}$

disc. 5

$$\int \frac{dx}{x^3-3x^2+4}$$

$$x^3 - 3x^2 + 4 = (x+1)(x^2+ax+b)$$

-1 is a root

$$= x^3 + ax^2 + bx + x^2 + ax + b$$

$$= x^3 + (a+1)x^2 + (a+b)x + b$$

$$b = 4 \qquad a+b=0 \quad \Rightarrow \quad a = -b = -4$$

$$x^3 - 3x^2 + 4 = (x+1)(x^2-4x+4) = (x+1)(x-2)^2$$

$$\begin{array}{r} x^2 - 4x + 4 \\ x+1 \;\big)\; x^3 - 3x^2 + 4 \\ -(x^3 + x^2) \\ -4x^2 \\ -(-4x^2 - 4x) \\ \hline 4x + 4 \\ -(4x+4) \\ \hline 0 \end{array}$$

$$\frac{1}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

↑ multiply by this

$$1 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

$$= A(x^2-4x+4) + B(x^2-x-2) + Cx + C$$

$$\Rightarrow 0 = \underbrace{(A+B)}_{=0}x^2 + \underbrace{(-4A-B+C)}_{=0}x + \underbrace{(4A-2B+C-1)}_{=0}$$

$$A = -B \qquad -4(-B) - B + C = 0 \qquad 4(-B) - 2B + C - 1 = 0$$

$$3B + C = 0 \qquad -6B + C = 1$$

$$C = -3B \qquad \Rightarrow \quad -6B - 3B = 1$$

$$\Rightarrow B = -1/9, \quad A = 1/9, \quad C = 1/3$$

$$\int \frac{dx}{(x+1)(x-2)^2} = \int \frac{1/9}{x+1} - \frac{1/9}{x-2} + \frac{1/3}{(x-2)^2}\,dx$$

$$= \frac{1}{9}\ln|x+1| - \frac{1}{9}|x-2| - \frac{1}{3}(x-2)^{-1} + c$$

$$\int \frac{100x}{(x-3)(x^2+1)^2}\,dx \qquad \frac{100x}{(x-3)(x^2+1)^2} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$100x = A(x^2+1)^2 + (Bx+C)(x^2+1)(x-3) + (Dx+E)(x-3)$$

$$\Rightarrow \underbrace{(A+B)}_{=0}x^4 + \underbrace{(C-3B)}_{=0}x^3 + \underbrace{(2A+B-3C+D)}_{=0}x^2 + (\underbrace{-3B+C+E-3D-100}_{=0})x + \underbrace{(A-3C-3E)}_{=0} = 0$$

$$A = -B \qquad C = 3B$$

$$\cancel{-3B} + \cancel{3B} + E - 3D - 100 = 0$$

$$-B - 9B - 3E = 0$$

$$E = -\frac{10}{3}B$$

eq. #3 : $10B - D = 0$

$$-10B - 9D - 300 = 0$$

$$D = -30$$

$$\Rightarrow \quad A = 3,\ B = -3,\ C = -9,\ D = -30,\ E = 10$$

$$\int \frac{100x}{(x-3)(x^2+1)^2}\,dx = \int \frac{3}{x-3} + \frac{-3x-9}{x^2+1} + \frac{-30x+10}{(x^2+1)^2}\,dx$$

$$= 3\int \frac{1}{x-3}\,dx - \underbrace{3\int \frac{x}{x^2+1}\,dx}_{} - 9\int \frac{1}{x^2+1}\,dx - \underbrace{30\int \frac{x}{(x^2+1)^2}\,dx}_{} + 10\int \frac{dx}{(x^2+1)^2}$$

$3\ln|x-3|$

$u = 1 + x^2$, $du = 2x\,dx$

$$\int \frac{\tfrac{1}{2}\,du}{u} = \frac{1}{2}\ln|x^2+1| + c$$

$-9\arctan x$

$u = 1 + x^2$, $du = 2x\,dx$

$$\int \frac{\tfrac{1}{2}\,du}{u^2} = -\frac{1}{2}(x^2+1)^{-1} + c$$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \int \cos^2\theta\,d\theta = \frac{1}{2}(\theta + \cos\theta\sin\theta) + c$$

$\tan^2\theta + 1 = \sec^2\theta$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}}\right) + c$$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + c$$

$$\int \frac{100x\,dx}{(x-3)(x^2+1)^2} = 3\ln|x-3| - \frac{3}{2}\ln|x^2+1| - 4\arctan x + \frac{15}{x^2+1} + \frac{5x}{x^2+1} + c$$

What are the possible orders for irreducible polynomials over $\mathbb{R}$?

Answer: only 1 or 2 (*) We will use the following:

1. Fundamental theorem of algebra

If P is a nonconstant polynomial with coefficients in $\mathbb{C}$, then P has a root in $\mathbb{C}$.

2. If z is a root of a polynomial P with real coefficients, then so is $\bar{z}$.

$z = a + bi \qquad \bar{z} = a - bi \qquad \overline{zw} = \bar{z}\,\bar{w}$

$\overline{z + w} = \bar{z} + \bar{w}$

$$z + \bar{z} = 2a, \quad z\bar{z} = a^2 + b^2$$

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0$$

$$\overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0} = \bar{0}$$

$$\overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0} = a_n \bar{z}^n + a_{n-1} \bar{z}^{n-1} + \cdots + a_0 = P(\bar{z})$$

Proof of (+): suppose P is a real polynomial of order ≥ 3.

If P has a real root a, then factor out $(x - a)$ $\longrightarrow$ done.

If P has no real roots,

let z be a complex root: this exists by FTA

By observation #2, $\bar{z}$ is also a root:

so we can factor out $(x - z)(x - \bar{z})$: we claim this is a real irreducible quadratic. To see this, write

$$(x - z)(x - \bar{z}) = x^2 - (\underbrace{z + \bar{z}}) x + \underbrace{z\bar{z}}$$

these are real