disc. 0

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Some review

Indefinite integration

We say F is an <u>antiderivative</u> of f if $F' = f$. We can write:

$$\int f(x)\,dx = F(x) + C$$

$$\int aspiri\; dn =$$

source: reddit

Ex. $\int x^3\,dx = \frac{1}{4}x^4 + C$

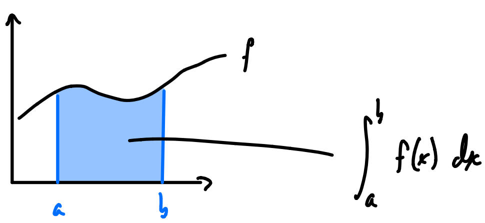
Ex. $\int \cos x\,dx = \sin x + C$

Ex. $\int ax\,dx = a\int x\,dx = a(x + C)$

$(ax + C)$

Definite integration

Geometric interpretation:



Integrals can be quite hard to compute $\left(\int_0^\infty \sin(x^2)\,dx = ?\right)$

The most important technique for computing them is the FTC:

<u>Thm.</u> If $F' = f$ and f is Riemann integrable on $[a, b]$, we have:

$$\int_a^b f(x)\,dx = \underbrace{F(b) - F(a)} = F(x)\Big|_a^b$$

Ex. $\int_{-1}^1 x^2\,dx = \frac{x^3}{3}\Big|_{-1}^1 = \frac{1^3}{3} - \frac{(-1)^3}{3} = \frac{2}{3}$

Ex. $\int_0^{\pi/4} \sec^2\theta\,d\theta = \tan\theta\Big|_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$

Exponentials (§7.1)

We know what x^3, $y^{1/2}$, $z^{3/7}$ mean. What about 2^π?

The exponential function "This is the most important function in mathematics" (Rudin)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{i.e. } \lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^n}{n!}$$

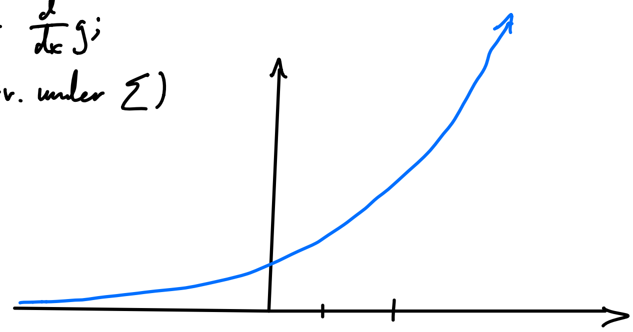
This is an infinite series (we'll see many more later)

The exponential fn. is its own derivative:

$$\frac{d}{dx} e^x = \frac{d}{dx}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\right) = 0 + 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$= e^x$$

(rec. $\frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g$;
for power series, can take deriv. under Σ)



It is positive and strictly increasing:
i.e. $e^x > 0$
& $x < y \Rightarrow e^x < e^y$

Its inverse fn. is called the natural logarithm:

$\ln : \mathbb{R}^+ \to \mathbb{R}$
$\{x \in \mathbb{R} : x > 0\}$

$$e^{x+y} = e^x e^y$$

$$\ln e^x = x$$

$$\ln a^b = b \ln a \qquad \ln xy = \ln x + \ln y$$

The number e is defined by

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718$$

How about other exponentials? Say $b > 0$. We can define

$$b^x = e^{x \ln b} = e^{(\ln b)x}$$

e.g. $\ln e = 1$, $e^x = e^{1\cdot x}$

Hence, $\frac{d}{dx} b^x = \frac{d}{dx} e^{(\ln b)x} = e^{(\ln b)x}(\ln b) = b^x \ln b$

e^x increases faster than any polynomial as $x \to \infty$:

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0$$

will prove this soon using L'Hôpital's rule.

Earlier: $\frac{d}{dx} x^n = n x^{n-1}$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

To see this, it's sufficient to show: $\frac{d}{dy} \ln y = \frac{1}{y}$.

Inverse function thm.:

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

In our example, $f(x) = e^x$, $f^{-1}(y) = \ln y$

$$\ln'(y) = \frac{1}{e^{\ln y}} = \frac{1}{y}$$

disc. 1

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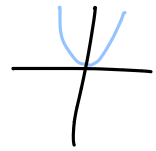
Some material from §7.2: preview for now.

$f:\mathbb{R}\to\mathbb{R}$ via $f(x)=x^2$



Definition of inverse

$f: D\to C$ has an inverse $f^{-1}: C\to D$ if $\underbrace{f\circ f^{-1}=id}_{f(f^{-1}(y))=y}$ and $\underbrace{f^{-1}\circ f=id}_{f^{-1}(f(x))=x}$.

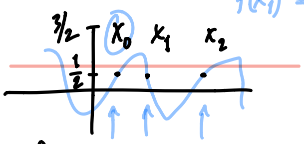
Injective & surjective functions $f: D\to C$

We say a function f is injective or one-to-one if $f(x)=f(y)\Rightarrow x=y$.

$f(x_1)=\frac{1}{2}\Rightarrow f^{-1}(f(x_1))=x_1$ so $f^{-1}(\frac{1}{2})=x_1$ $\quad$ $f(f^{-1}(3/2)) = 3/2$

$f^{-1}(1/2)$

$f^{-1}(f(x))=x$

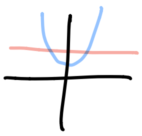


horizontal line test: line intersects the graph of f at most once.

We say f is surjective or onto if for every $y\in C$, there exists $x\in D$ such that $f(x)=y$

The range of f is the set $f(D)=\{f(x): x\in D\}$. In particular f is onto if and only if its range is equal to its codomain.

Ex. $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2+1$. This function is not injective and not surjective



range is $\{x\in\mathbb{R}: x\geq 1\}\neq\mathbb{R}$

Consider $\tilde{f}:\mathbb{R}\to\{x\in\mathbb{R}: x\geq 1\}$ via $\tilde{f}(x)=x^2+1$: this is surjective.

Or, $g:\mathbb{R}^+\to\mathbb{R}$ via $g(x)=x^2+1$: this is injective & not surjective.

$\mathbb{R}^+ = \{x\in\mathbb{R}: x>0\}$

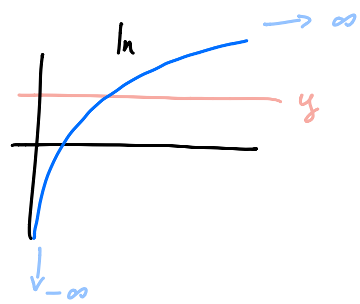
Defining an inverse

A function f is invertible iff. it is both injective and surjective. "bijective"

For $y\in C$, set

$f^{-1}(y)=$ the unique x s.t. $f(x)=y$

(so $f^{-1}(f(x))=x$, $f(f^{-1}(y))=y$)



Example: exponential $\quad e^x:\mathbb{R}\to\mathbb{R}^+$ has an inverse $\ln:\mathbb{R}^+\to\mathbb{R}$

Inverse function thm. Suppose f is invertible & differentiable, $y\in$ domain f^{-1}, and $f'(f^{-1}(y))\neq 0$.

is in

Then $(f^{-1})'(y)=\dfrac{1}{f'(f^{-1}(y))}$

Ex. $f: \mathbb{R} \to \mathbb{R}^+$ via $f(x) = e^x$, $f^{-1}: \mathbb{R}^+ \to \mathbb{R}$ via $f^{-1}(x) = \ln x$

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f(f^{-1}(y))} = \frac{1}{y}$$

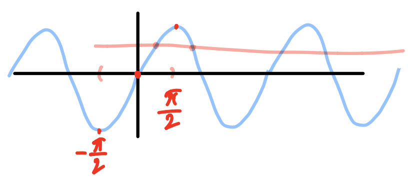
Ex. 1

a. §7.2#3: What is the largest interval containing zero on which $f(x) = \sin x$ $: \mathbb{R} \to \mathbb{R}$ is one-to-one?

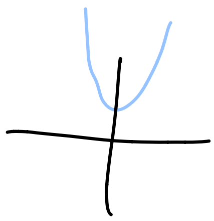
b. §7.2#2: Is $f(x) = x^2 + 2$ $f: \mathbb{R} \to \mathbb{R}$ one-to-one? If not, describe a domain on which it is one-to-one.

$=$ does f have an inverse? On what domain?

a.



b. $ax^2 + bx + c$, $a \neq 0$



$\{x : x > 0\}$

$f: \mathbb{R}^+ \to \{x : x > 2\}$ does have an inverse

$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$

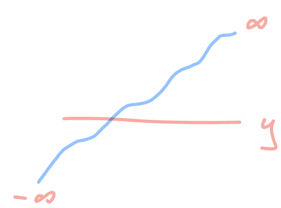
Ex. 2 Let $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be defined by $f(x) = 2\tan x$. What is f^{-1}?

$y = 2\tan x \Rightarrow \frac{y}{2} = \tan x \Rightarrow \arctan\frac{y}{2} = x$

$f^{-1}(y) = \arctan\frac{y}{2}$

Ex. 3 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 2\sin x$.

(a) Show that f has an inverse.



1-1: note that $f'(x) = 3 + 2\cos x \geq 3 + 2(-1) = 1$

let $x < y$. By the MVT, $\frac{f(y) - f(x)}{y - x} = f'(c) \geq 1$ for some $c \in (x, y)$

$\Rightarrow f(y) - f(x) \geq y - x > 0 \Rightarrow f(x) < f(y)$

$f(x) \neq f(y)$

surjective: note $f \to \infty$ as $x \to \infty$

$f \to -\infty$ as $x \to -\infty$

By continuity & IVT, f is surjective

(b) Find $(f^{-1})'(0)$.

disc. 1 (1F)

– 10:30

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Some material from §7.2:

Definition of inverse

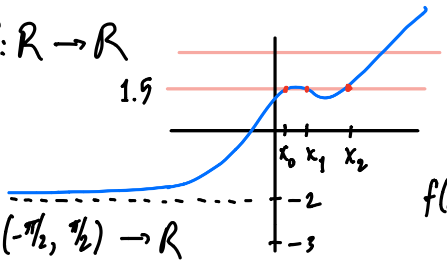
If $f: D \to C$, we say f has an inverse $f^{-1}: C \to D$ if $\underbrace{f \circ f^{-1} = id}_{f(f^{-1}(y)) = y}$, $\underbrace{f^{-1} \circ f = id}_{f^{-1}(f(x)) = x}$

Injective & surjective functions $f: D \to C$

We say f is injective or one-to-one if $f(x) = f(y) \Rightarrow x = y$

$x \neq y \Rightarrow f(x) \neq f(y)$ e.g. f strictly increasing
$x < y \Rightarrow f(x) < f(y)$

$f: R \to R$



$f: (-\pi/2, \pi/2) \to R$

(horizontal line test)

not injective. $\underbrace{f(f^{-1}(-3))}_{\geq -2} = -3$ $f^{-1}(f(x)) = x$ $f^{-1}(f(x_1)) = x_1$

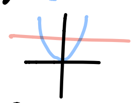
$f(x_0) = f(x_1) = f(x_2) = 1.5$ $x_1 = \cancel{f^{-1}(1.5)} = x_0$

We say f is surjective or onto if for every $y \in C$ (→ is an element of), there exists $x \in D$ such that $f(x) = y$.

The range of f is the set $R(f) = \{f(x) : x \in D\}$. So f is surjective if and only if $R(f) = C$.

Ex. $f: R \to R$ defined by $f(x) = x^2$ $\{x : x \geq 0\} \cup \{-1\}$, R $\subseteq$

(i) Not injective



(ii) Not surjective, since $f \geq 0$

$g: R \to \{x \in R : x \geq 0\}$ defined by $g(x) = x^2$
This function is surjective.
Given $y \geq 0$, $y = (\sqrt{y})^2 = f(\sqrt{y})$

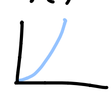
Defining an inverse $f: D \to C$

A function is invertible if and only if it is $\underbrace{\text{injective \& surjective.}}_{\text{i.e. bijective}}$

For $y \in C$, $\underset{x}{f^{-1}(y)}$ = the unique x such that $f(x) = y$
If $f(x_1) = y = f(x_2)$, then $x_1 = x_2$
$f(f^{-1}(y)) = f(x) = y$, $f^{-1}(f(x)) = f^{-1}(y) = x$

$f: \{x : x \geq 0\} \to R$

$f(x) = x^2$



$f: \{x : x \geq 0\} \to \{x : x \geq 0\}$

Example: exponential $e^x : R \to R^+$ has an inverse $\ln : R^+ \to R$

$R^+ \downarrow \{x \in R : x > 0\}$

Inverse function thm. Assume f is differentiable & invertible, $y \in$ domain of f^{-1}, $f'(f^{-1}(y)) \neq 0$.

Then

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

Ex. $f(x) = e^x$, $f^{-1}(y) = \ln y$,

$$\frac{d}{dy}\ln y = \frac{1}{f'(\ln y)} = \frac{1}{e^{\ln y}} = \frac{1}{y}$$

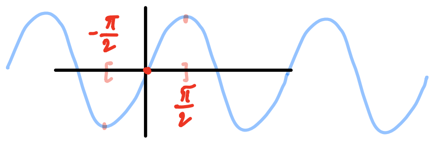
Ex. 1

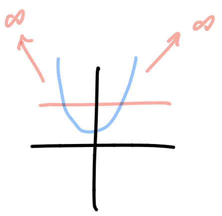
a. §7.2#3: What is the largest interval containing zero on which $f(x) = \sin x$ $: \mathbb{R} \to \mathbb{R}$ is one-to-one?

b. §7.2#2: Is $f(x) = x^2 + 2$ one-to-one? If not, describe a domain on which it is one-to-one.

$=$ $f: \mathbb{R} \to \mathbb{R}$

Does f have an inverse? On what domain?

a. 

b. 

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = ax^2 + bx + c$ $\quad a \neq 0$

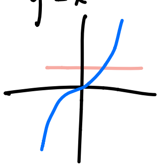
not injective

$g: \mathbb{R} \to \mathbb{R}$

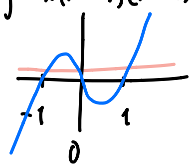
$g(x) = ax + b$ $\quad a \neq 0$

is injective

$y = x^3$



$y = x(x-1)(x+1)$



$ax^3 + bx^2 + cx + d$: depends

Ex. 2 Let $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be defined by $f(x) = 2\tan x$. What is f^{-1}?

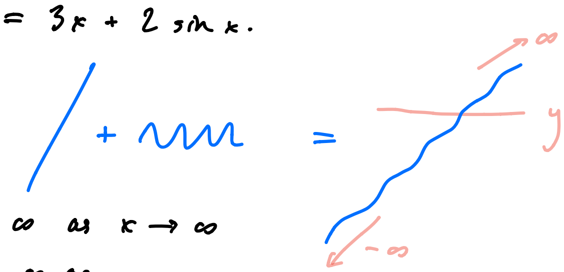
$y = 2\tan x$ $\Rightarrow$ $\frac{y}{2} = \tan x$ $\Rightarrow$ $x = \arctan\frac{y}{2}$

solve for x in terms of y

$f^{-1}(y) = \arctan\frac{y}{2}$

Ex. 3 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 2\sin x$.

(a) Show that f has an inverse.



f surjective: note that: $\begin{cases} f \to \infty \text{ as } x \to \infty \\ f \to -\infty \text{ as } x \to -\infty \\ f \text{ is continuous} \end{cases}$

By the intermediate value theorem, there exists x s.t. $f(x) = y$

f injective: claim f strictly increasing

$$f'(x) = 3 + 2\cos x \geq 3 + 2(-1) = 1$$

≥ -1

If $x < y$, by the mean value there is a c $\in (x, y)$ s.t.

$$\frac{f(y) - f(x)}{y - x} = f'(c) \geq 1$$

[(b) Find $(f^{-1})'(0)$.]

$$\Rightarrow f(y) - f(x) \geq y - x > 0$$

$$\Rightarrow f(x) < f(y)$$

disc. 2 (1E)

Another interesting approach to the logarithm (from Rogawski §7.3 Ex. 116)

Define a function G for $x > 0$:

$$G(x) = \int_1^x \frac{1}{t}\,dt$$

116. Defining $\ln x$ as an Integral This exercise proceeds as if we didn't know that $G(x) = \ln x$ and shows directly that G has all the basic properties of the logarithm. Prove the following statements.

(a) $\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{t}\,dt$ for all $a, b > 0$. Hint: Use the substitution $u = t/a$.

$u = t/a \downarrow dt = a\,du$

$$\int_1^b \frac{1}{au}\,a\,du = \int_1^b \frac{du}{u}$$

(b) $G(ab) = G(a) + G(b)$. Hint: Break up the integral from 1 to ab into two integrals and use (a).

$$G(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t}$$

$$= G(a) + G(b)$$

(c) $G(1) = 0$ and $G(a^{-1}) = -G(a)$ for $a > 0$.

$\int_1^1 \frac{dt}{t}$

$$0 = G(1) = G(aa^{-1}) = G(a) + G(a^{-1}) \Rightarrow G(a^{-1}) = -G(a).$$

(d) $G(a^n) = nG(a)$ for all $a > 0$ and integers n. (clear for $n = 0$)

if $n > 0$: $G(aa^{n-1}) = G(a) + G(a^{n-1}) = G(a) + G(aa^{n-2}) = G(a) + (G(a) + G(a^{n-2}))$
$= \cdots = nG(a)$

if $n < 0$: $G(a^n) = G((a^{-1})^{-n}) \underset{\substack{\uparrow \\ \text{first part}}}{=} -n\,G(a^{-1}) \underset{\substack{\uparrow \\ (c)}}{=} -n(-G(a)) = nG(a).$

(e) $G(a^{1/n}) = \frac{1}{n}G(a)$ for all $a > 0$ and integers $n \neq 0$.

We have $G(a) = G((a^{1/n})^n) = n\,G(a^{1/n}) \Rightarrow G(a^{1/n}) = \frac{1}{n}G(a)$

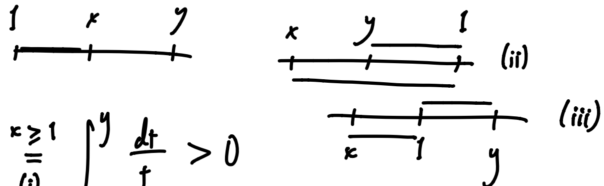
$a = \underbrace{a^{1/n}a^{1/n}\cdots a^{1/n}}_{n \text{ times}}$

(f) $G(a^r) = rG(a)$ for all $a > 0$ and rational numbers r.

Say $r = p/q$ where p, q are integers with $q \neq 0$. Combining (d) & (e):

$$G(a^r) = G(a^{p/q}) = G((a^{1/q})^p) \overset{(d)}{=} p\,G(a^{1/q}) \overset{(e)}{=} \frac{p}{q}G(a) = rG(a)$$

(g) G is increasing. ~~[Hint: Use FTC II.]~~



$$x < y \Rightarrow G(y) - G(x) = \int_1^y \frac{dt}{t} - \int_1^x \frac{dt}{t} \underset{(i)}{\overset{x \geq 1}{=}} \int_x^y \frac{dt}{t} > 0$$

$$\underset{(ii)}{\overset{y \leq 1}{=}} -\int_y^1 + \int_x^1 = \int_x^y > 0 \qquad (iii) \overset{x \leq 1 < y}{=} \int_1^y + \int_x^1 = \int_x^y \frac{dt}{t} > 0$$

(h) There exists a number a such that $G(a) > 1$. Hint: Show that $G(2) > 0$ and take $a = 2^m$ for $m > 1/G(2)$. $\longrightarrow$ $mG(2) > 1$

Note that $G(2) = \int_1^2 \frac{dt}{t} > 0$. Moreover,

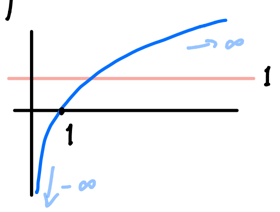
$$G(a) = G(2^m) = mG(2) > 1$$

(i) $\lim_{x\to\infty} G(x) = \infty$ and $\lim_{x \to 0^+} G(x) = -\infty$.

$$\lim_{m\to\infty} G(2^m) = \lim_{m\to\infty} mG(2) = \underbrace{G(2)}_{>0} \lim_{m\to\infty} m = \infty$$

$$\lim_{m\to\infty} G(2^{-m}) = \lim_{m\to\infty} -mG(2) = -\infty$$

(j) There exists a unique number E such that $G(E) = 1$.



G strictly increasing $\Rightarrow$ G one-to-one

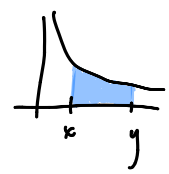
(i) $\Rightarrow$ E exists, using intermediate value thm.

G is continuous:

$$G(y) - G(x) = \int_x^y \frac{dt}{t}$$

as $y \searrow x$

$$G(y) - G(x) \longrightarrow 0$$



(k) $G(E^r) = r$ for every rational number r.

$$G(E^r) = rG(E) = r$$

So we can define $\ln x = G(x)$.

What can we do with logarithms?

$$\underbrace{\log xy = \log x + \log y}$$

They convert multiplication into addition, which can be useful.

__Ex.__ Find the derivative of $f(x) = e^{-x^2}(x^3+1)^{1/5}\cos x^6$

Note: using the chain rule, $\frac{d}{dx}\ln f(x) = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)} \Rightarrow f' = f(\ln f)'$

$$(\ln f)' = \frac{d}{dx}\left(\ln e^{-x^2} + \ln (x^3+1)^{1/5} + \ln \cos x^6\right)$$

$$= \frac{d}{dx}\left(-x^2 + \frac{1}{5}\ln(x^3+1) + \ln\cos x^6\right)$$

$$= -2x + \frac{1}{5}\frac{3x^2}{x^3+1} + \frac{1}{\cos x^6}(-\sin x^6 \cdot 6x^5)$$

$$\Rightarrow f' = e^{-x^2}(x^3+1)^{1/5}\cos x^6\left(-2x + \frac{3x^2}{5(x^3+1)} - 6x^5\tan x^6\right)$$

E.g. $(\underbrace{p_1(x)\cdots p_n(x)}_{f})' = 0$ instead look at $\ln(p_1\cdots p_n) = \underbrace{\ln p_1 + \cdots + \ln p_n}$

$f(x^*) \geq f(x) \Rightarrow \ln f(x^*) \geq \ln f(x)$

Exponential growth/decay

$$\begin{cases} y' = ky \\ y(0) = C \end{cases}$$ growth if $k>0$, decay if $k<0$

$$y(t) = Ce^{kt}$$

Doubling time / half-life & generalizations

$t_1 < t_2$ $\qquad y(t_2) = \alpha y(t_1) \qquad t_2 - t_1$

$$\cancel{C}e^{kt_2} = \alpha \cancel{C}e^{kt_1}$$

$$kt_2 = \ln \alpha e^{kt_1} = \ln\alpha + kt_1$$

$$\Rightarrow k(t_2 - t_1) = \ln\alpha$$

$$\Rightarrow t_2 - t_1 = \frac{\ln\alpha}{k}$$

e.g. half-life is $\frac{\ln 1/2}{k}$

disc. 2 (1F)

Another interesting approach to the logarithm (from Rogawski §7.3 Ex. 116)

Define a function G for $x > 0$:

$$G(x) = \int_1^x \frac{1}{t}\,dt$$

116. Defining $\ln x$ as an Integral This exercise proceeds as if we didn't know that $G(x) = \ln x$ and shows directly that G has all the basic properties of the logarithm. Prove the following statements.

(a) $\displaystyle\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{t}\,dt$ for all $a, b > 0$. Hint: Use the substitution $u = t/a$.

$u = t/a \quad \| \quad du = dt/a$

$t = a \Rightarrow u = 1$

$t = ab \Rightarrow u = b$

$$\int_1^b \frac{1}{\cancel{a}u}\,\cancel{a}\,du = \int_1^b \frac{du}{u}.$$

(b) $G(ab) = G(a) + G(b)$. Hint: Break up the integral from 1 to ab into two integrals and use (a).

$$G(ab) = \int_1^{ab} \frac{dt}{t} = \underbrace{\int_1^a \frac{dt}{t}}_{G(a)} + \underbrace{\int_a^{ab} \frac{dt}{t}}_{\xrightarrow{\text{by (a)}} \int_1^b \frac{dt}{t}} = G(a) + G(b).$$

(c) $G(1) = 0$ and $G(a^{-1}) = -G(a)$ for $a > 0$.

$$G(1) = \int_1^1 \frac{dt}{t} = 0 \qquad 0 = G(1) = G(aa^{-1}) = G(a) + G(a^{-1})$$

$$\Rightarrow G(a^{-1}) = -G(a)$$

(d) $G(a^n) = nG(a)$ for all $a > 0$ and integers n. (clear for $n = 0$)

$n > 0$: $G(a^n) = G(aa^{n-1}) = G(a) + G(a^{n-1}) = G(a) + G(a) + G(a^{n-2}) \quad \ldots$

$= nG(a)$

$n < 0$: $G(a^n) = G((a^{-1})^{-n}) = -nG(a^{-1}) = -n(-G(a)) = nG(a).$

(e) $G(a^{1/n}) = \frac{1}{n}G(a)$ for all $a > 0$ and integers $n \neq 0$.

$$G(a) = G(\underbrace{a^{1/n} \cdots a^{1/n}}_{n \text{ times}}) = G((a^{1/n})^n) = nG(a^{1/n})$$

$$\Rightarrow G(a^{1/n}) = \frac{1}{n}G(a)$$

(f) $G(a^r) = rG(a)$ for all $a > 0$ and rational numbers r.

Say $r = p/q$ where p, q are integers & $q \neq 0$

$$G(a^r) = G(a^{p/q}) = G((a^{1/q})^p) \overset{(d)}{=} pG(a^{1/q}) \overset{(e)}{=} p \cdot \frac{1}{q} G(a) = rG(a).$$

(g) G is increasing. [Hint: Use FTC II.]

$$x < y \Rightarrow G(y) - G(x) = \int_1^y \frac{dt}{t} - \int_1^x \frac{dt}{t} = \int_x^y \underbrace{\frac{1}{t}}_{>0} dt > 0$$

$$\Rightarrow G(x) < G(y).$$

Note: $G'(x) = \frac{d}{dx}\int_1^x \frac{dt}{t} = \frac{1}{x} > 0$ so in particular, G is continuous

(h) There exists a number a such that $G(a) > 1$. Hint: Show that $G(2) > 0$ and take $a = 2^m$ for $m > 1/G(2)$. $mG(2) > 1$

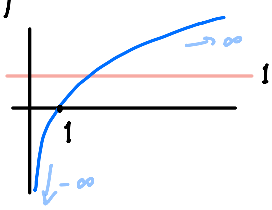
Note $\int_1^2 \frac{dt}{t} > 0$, so $G(a) = mG(2) > 1$.

(i) $\lim_{x\to\infty} G(x) = \infty$ and $\lim_{x\to 0^+} G(x) = -\infty$. (Note that G is increasing.)

$$\lim_{m\to\infty} G(2^m) = \lim_{m\to\infty} mG(2) = \underbrace{G(2)}_{>0} \lim_{m\to\infty} m = \infty$$

$$\lim_{m\to\infty} G(2^{-m}) = \lim_{m\to\infty} -mG(2) = -G(2)\lim_{m\to\infty} m = -\infty$$

(j) There exists a unique number E such that $G(E) = 1$.



Since G is cont. & increasing, by the IVT there is an E s.t. $G(E) = 1$; this E is unique.

(k) $G(E^r) = r$ for every rational number r.

$$G(E^r) = rG(E) = r$$

What can we do with logarithms?

$\log(xy) = \log x + \log y$

They convert multiplication into addition, which can be useful.

$\log(xyz) = \log x + \log y + \log z$

Ex. Find the derivative of $f(x) = e^{-x^2}(x^3+1)^{1/5}\cos x^6$

Key observation $\frac{d}{dx}\ln f(x) = \frac{1}{f(x)}\cdot f'(x) = \frac{f'(x)}{f(x)}$, so $f' = f(\ln f)'$

$$(\ln f(x))' = \frac{d}{dx}\left(\ln e^{-x^2} + \ln(x^3+1)^{1/5} + \ln\cos x^6\right)$$

$$= \frac{d}{dx}\left(-x^2 + \frac{1}{5}\ln(x^3+1) + \ln\cos x^6\right)$$

$(\cos x^6)' = -\sin x^6 \cdot 6x^5$

$$= -2x + \frac{1}{5}\frac{1}{x^3+1}\cdot 3x^2 + \frac{1}{\cos x^6}(-\sin x^6\cdot 6x^5)$$

$$\Rightarrow f'(x) = e^{-x^2}(x^3+1)^{1/5}\cos x^6\left(-2x + \frac{3x^2}{5(x^3+1)} - 6x^5\tan x^6\right).$$

E.g. trying to minimize/maximize $p_1(x)\cdots p_n(x)$.

$(p_1\cdots p_n)' = 0$

Equivalent to minimize $\ln(p_1\cdots p_n) = \ln p_1 + \cdots + \ln p_n$

Exponential growth/decay

$$\begin{cases} y'(t) = ky(t) \\ y(0) = C \end{cases}$$

growth if $k>0$

$$\underline{y(t) = Ce^{kt}}$$

Doubling time / half-life & generalizations

$t_1 < t_2$ Q: what is $t_2 - t_1$?

$y(t_1)$ things $\quad$ $\alpha\cdot y(t_1)$ things $(\alpha > 0)$

$$y(t_2) = \alpha y(t_1) \Rightarrow \cancel{C}e^{kt_2} = \alpha\cancel{C}e^{kt_1}$$

$$\overset{\ln}{\Rightarrow} kt_2 = \ln(\alpha e^{kt_1}) = \ln\alpha + kt_1$$

$$\Rightarrow t_2 - t_1 = \frac{\ln\alpha}{k}$$

E.g. half-life is $\frac{\ln 1/2}{k}$

disc. 3 (1E)

L'Hôpital's rule

Suppose f, g are differentiable on (a, b), where $a < b$, and $c \in (a, b)$, $g'(x) \neq 0$ for x near c (except perhaps at $x = c$).

If (i) $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$

or

(ii) $\lim_{x \to c} g(x) = \infty$

then if $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists, we have

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

(The rule also works for one-sided limits, or limits at ∞).

$$\left\{ \begin{array}{ll} \lim_{x \to \infty} \frac{e^x}{x^2} & \frac{\infty}{\infty} \\ \lim_{x \to \infty} \frac{(\sin x)/x^2}{\underbrace{1/x^2}_{\sin x}} & \frac{0}{0} \end{array} \right.$$

Ex. 1 $\lim_{x \to \infty} \frac{x^2}{e^x} \overset{L'H}{=} \lim_{x \to \infty} \frac{2x}{e^x} \overset{L'H}{=} \lim_{x \to \infty} \frac{2}{e^x} = 0$

$2\underbrace{\left(\frac{1}{e}\right)^x}$ $\qquad c^x \qquad 0 < c < 1$

Ex. 2 $\lim_{x \to 0^+} x^x$ $\qquad \ln x^x = x \ln x = \frac{\ln x}{1/x} \qquad \frac{-\infty}{\infty}$

$$\lim_{x \to 0} \frac{\ln x}{1/x} \overset{L'H}{=} \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \to 0} \frac{-x^2}{x} = \lim_{x \to 0} x = 0$$

i.e. $\lim_{x \to 0} x \ln x = 0$

f continuous $\Rightarrow \lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$

$$\Rightarrow \lim_{x \to 0} x^x = \lim_{x \to 0} e^{x \ln x} = e^{\lim_{x \to 0} x \ln x} = e^0 = 1$$

Note: Möbius claimed if $f(x) \to 0$, $g(x) \to 0$ as $x \to 0^+$, then $f(x)^{g(x)} \to 1$ as $x \to 0^+$

Ex. Show that this is false.

Ex. 3 $\lim_{x \to \infty} \overset{\lim f/g}{\frac{\log x + \sin x}{x}} = 0 \neq \lim_{x \to \infty} \overset{\lim f'/g'}{\frac{\frac{1}{x} + \cos x}{1}}$ DNE

$\|$

$\underline{\lim_{x \to \infty} \frac{\log x}{x}} + \underline{\lim_{x \to \infty} \frac{\sin x}{x}}$ $\qquad |\sin x| \leq 1 \qquad 0 \leq \left|\frac{\sin x}{x}\right| \leq \frac{1}{x}$ as $x \to \infty$

$= \lim_{x \to \infty} \frac{1/x}{1} = 0$

Ex. 4 $\displaystyle\lim_{x\to0^+}\frac{e^{-1/x}}{x}$

$$\lim_{x\to0^+}\frac{\frac{d}{dx}e^{-1/x}}{\frac{d}{dx}x} = \lim_{x\to0^+}\frac{e^{-1/x}\left(\frac{1}{x^2}\right)}{1} = \lim_{x\to0^+}\frac{e^{-1/x}}{x^2} \quad ??$$

$$\| $$

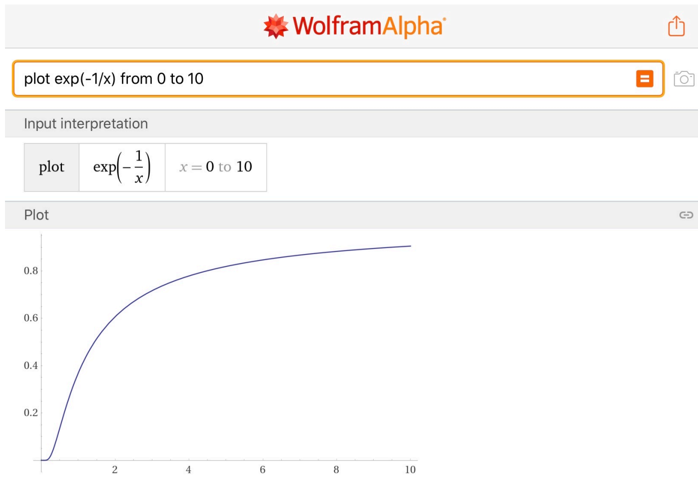
$$\lim_{x\to0^+}\frac{1/x}{e^{1/x}} \overset{L'H}{=} \lim_{x\to0^+}\frac{-1/x^2}{e^{1/x}\left(-1/x^2\right)} = \lim_{x\to0^+} e^{-1/x} = 0$$

Application: an interesting function

$$f(x) = \begin{cases} e^{-1/x} & x>0 \\ 0 & x\le 0\end{cases}$$

$$f'(0)\ ?\ \begin{cases}\displaystyle\lim_{x\to0^+}\frac{e^{-1/x}-0}{x-0} = \lim_{x\to0^+}\frac{e^{-1/x}}{x} = 0 \\ \displaystyle\lim_{x\to0^-}\frac{0-0}{x-0} = 0\end{cases}$$

$$\Rightarrow f'(0) = 0.$$



Inverse trig. functions

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

$$\frac{d}{dy}\arccos y \overset{\text{inv. fn. thm.}}{=} \frac{1}{\cos'(\arccos y)} = \frac{-1}{\sin(\arccos y)} = \frac{-1}{\sqrt{1-y^2}} \quad (|y|<1)$$

$$\frac{d}{dy}\operatorname{arcsec} y = \frac{1}{\sec'(\operatorname{arcsec} y)} = \frac{1}{\underbrace{\sec \operatorname{arcsec} y}_{y}\cdot\tan\operatorname{arcsec} y}$$

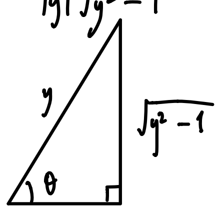
$$= \begin{cases}\dfrac{1}{y\sqrt{y^2-1}} & \text{if } y>1 \\ \dfrac{1}{y\left(-\sqrt{y^2-1}\right)} & \text{if } y<-1\end{cases}$$

$$= \frac{1}{|y|\sqrt{y^2-1}}$$

$$\frac{d}{dy}\operatorname{arccsc} y \overset{\text{sim.}}{=} \frac{-1}{|y|\sqrt{y^2-1}}$$

$$\frac{d}{dy}\operatorname{arccot} y$$

$y>1$:

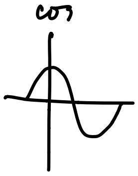


$$\sec\theta = \frac{y}{1}$$

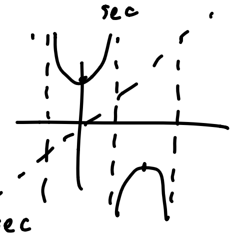
$$\theta = \operatorname{arcsec} y$$

$$\tan\operatorname{arcsec} y = \tan\theta = \frac{\sqrt{y^2-1}}{1}$$

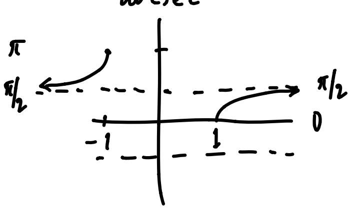
cos



sec



arcsec



Recall:

$$\frac{d}{dx}\sec x = \sec x\tan x$$

$$\frac{d}{dx}\csc x = -\csc x\cot x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

arcsin : $[-1,1] \to [-\frac{\pi}{2},\frac{\pi}{2}]$

arccos : $[-1,1] \to [0,\pi]$

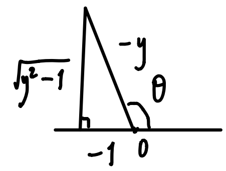
arctan : $\mathbb{R} \to (-\pi/2, \pi/2)$

arccot : $\mathbb{R} \to (0,\pi)$

arcsec : $\{|x|\ge1\} \to [0,\pi]\setminus\{\pi/2\}$

arccsc : $\{|x|\ge1\} \to [-\frac{\pi}{2},\frac{\pi}{2}]\setminus\{0\}$

$y < -1$:

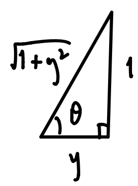


$$\sec\theta = \frac{-y}{-1} = y$$

$$\theta = \operatorname{arcsec} y$$

$$\tan \operatorname{arcsec} y = \tan\theta = \frac{\sqrt{y^2-1}}{-1}$$

$$\frac{d}{dy} \operatorname{arccot} y = \frac{-1}{(\csc(\operatorname{arccot} y))^2}$$



$$\cot\theta = \frac{y}{1} = y$$

$$\Rightarrow \theta = \operatorname{arccot} y$$

$$\csc \operatorname{arccot} y = \csc\theta = \frac{\sqrt{1+y^2}}{1}$$

$$= \frac{-1}{(\sqrt{1+y^2})^2}$$

$$= \frac{-1}{1+y^2}.$$

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SMC Wed 12PM 937 7987 3834

disc. 3 (1F)

L'Hôpital's rule

Suppose f, g are differentiable on (a, b), where $a < b$, and $c \in (a, b)$, $g'(x) \neq 0$ for x near c (except perhaps at $x = c$).

If (i) $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$

or

(ii) $\lim_{x \to c} g(x) = \infty$

then if $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists, we have

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

(The rule also works for one-sided limits, or limits at ∞).

Ex. 1 $\lim_{x \to \infty} \frac{x^2}{e^x} \overset{L'H}{=} \lim_{x \to \infty} \frac{2x}{e^x} \overset{L'H}{=} \lim_{x \to \infty} \frac{2}{e^x} = 0$

Ex. 2 $\lim_{x \to 0^+} x^x$ Take a log & use continuity

$\ln(x^x) = \underset{0 \cdot (-\infty)}{x \ln x} = \frac{\ln x}{1/x}$ $\lim_{x \to 0^+} \frac{\ln x}{1/x} \overset{L'H}{=} \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$

Hence $\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^{\lim_{x \to 0^+} x \ln x} = e^0 = 1$

Note: Möbius claimed $f(x) \to 0$, $g(x) \to 0 \Rightarrow f(x)^{g(x)} \to 1$

But this claim is false.

Ex. 3 $\lim_{x \to \infty} \frac{\log x + \sin x}{x} = 0 \neq \lim_{x \to \infty} \frac{1/x + \cos x}{1} = \lim_{x \to \infty} \left(\frac{1}{x} + \cos x\right)$ DNE

$= \underbrace{\lim_{x \to \infty} \frac{\log x}{x}} + \underbrace{\lim_{x \to \infty} \frac{\sin x}{x}}$

$\overset{L'H}{=} \lim_{x \to \infty} \frac{1/x}{1} = 0$ $\left|\frac{\sin x}{x}\right| \leq \frac{1}{x}$

Ex. 4 $\lim_{x\to 0^+} \frac{e^{-1/x}}{x}$ $\qquad \lim_{x\to 0^+} \frac{e^{-1/x}\left(1/x^2\right)}{1} = \lim_{x\to 0^+} \frac{e^{-1/x}}{x^2}$

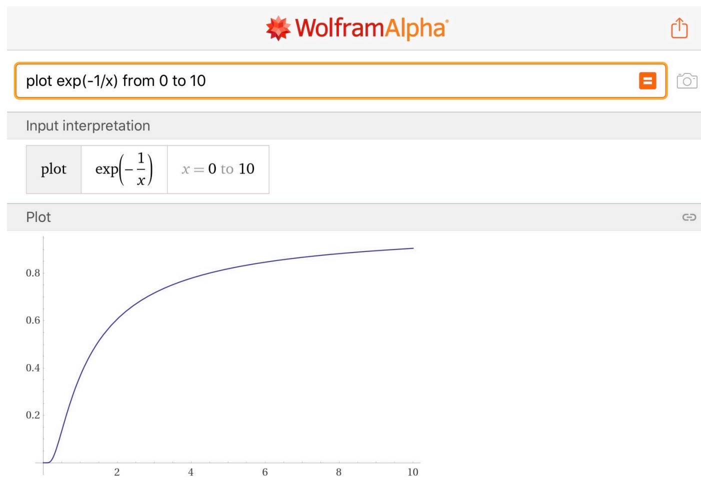
Note: $x \to 0^+ \Rightarrow \frac{-1}{x} \to -\infty \Rightarrow e^{-1/x} \to 0$

$$\parallel$$

$$\lim_{x\to 0^+} \frac{1/x}{e^{1/x}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\cancel{-1/x^2}}{e^{1/x}\left(\cancel{-1/x^2}\right)} = \lim_{x\to 0^+} e^{-1/x} = 0$$

Application: an interesting function

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0 \end{cases} \qquad f'(0)?$$



$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \to \lim_{x\to 0^+} \frac{e^{-1/x}}{x} = 0$$

$$\to \lim_{x\to 0^-} \frac{0}{x} = 0$$

$$f'(x) = \begin{cases} e^{-1/x}/x^2 & x > 0 \\ 0 & x \le 0 \end{cases}$$

$$\lim_{x\to 0^+} \frac{e^{-1/x}}{x^3} = \lim_{x\to 0^+} \frac{x^{-3}}{e^{1/x}} = \lim_{x\to 0^+} \frac{-3\cancel{x^{-4}}}{-e^{1/x}\cancel{x^{-2}}} = 0$$

$$f^{(n)}(0) = 0$$

Inverse trig functions

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

$(|y| < 1)$ $\quad \frac{d}{dy} \arccos y = \frac{1}{\cos'(\arccos y)} = \frac{1}{-\sin(\arccos y)} = \frac{-1}{\sqrt{1-y^2}}$

$(|y| > 1)$ $\quad \frac{d}{dy} \operatorname{arcsec} y = \frac{1}{\sec'(\operatorname{arcsec} y)} = \frac{1}{\underbrace{\sec \operatorname{arcsec} y}_{y}\ \underbrace{\tan \operatorname{arcsec} y}}$

$$= \begin{cases} \frac{1}{y\sqrt{y^2-1}} & y > 1 \\ \frac{1}{y\left(-\sqrt{y^2-1}\right)} & y < -1 \end{cases}$$

$$= \frac{1}{|y|\sqrt{y^2-1}}$$

$(|y| > 1)$ $\quad \frac{d}{dy} \operatorname{arccsc} y = \frac{-1}{|y|\sqrt{y^2-1}}$

$y \in \mathbb{R}$ $\quad \frac{d}{dy} \operatorname{arccot} y$

Recall:

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

arcsin : $[-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

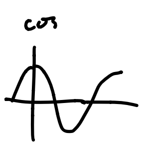
arccos : $[-1, 1] \to [0, \pi]$

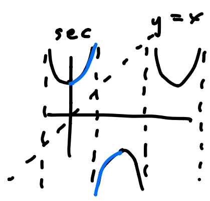
arctan : $\mathbb{R} \to (-\pi/2, \pi/2)$

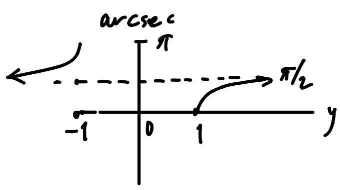
arccot : $\mathbb{R} \to (0, \pi)$

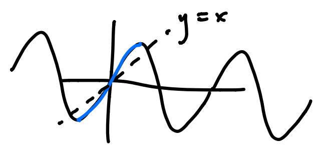
arcsec : $\{|x| \ge 1\} \to [0, \pi] \setminus \{\pi/2\}$

arccsc : $\{|x| \ge 1\} \to [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

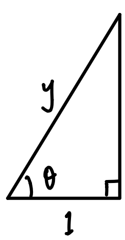








$y > 1$



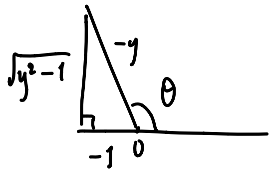
$\sqrt{y^2-1}$

$\operatorname{arcsec} y = \theta$

$y = \sec \theta$

$\tan \operatorname{arcsec} y = \tan \theta = \frac{\sqrt{y^2-1}}{1}$

$y < -1$

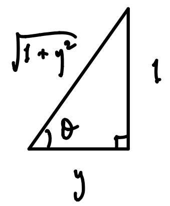


$\operatorname{arcsec} y = \theta$

$\sec\theta = \frac{-y}{-1} = y$

$\tan \operatorname{arcsec} y = \frac{\sqrt{y^2 - 1}}{-1}$

$$\frac{d}{dy} \operatorname{arccot} y = \frac{1}{-\csc^2 \operatorname{arccot} y} = \frac{-1}{(\csc \operatorname{arccot} y)^2} = \frac{-1}{(\sqrt{1 + y^2})^2} = \frac{-1}{1 + y^2}$$



$\operatorname{arccot} y = \theta$

$\cot\theta = y$

$\csc \operatorname{arccot} y = \csc\theta = \frac{\sqrt{1 + y^2}}{1}$

disc. 4 (1E)

Today: • some examples of integration by parts

• review problems from the book

Integration by parts $\quad u(b)v(b) - u(a)v(a) = \int_a^b (uv)'(t)\,dt$

$$= \int_a^b u'(t)v(t)\,dt + \int_a^b u(t)v'(t)\,dt$$

$$\Rightarrow \int_a^b u(t)v'(t)\,dt = u(t)v(t)\Big|_a^b - \int_a^b u'(t)v(t)\,dt \qquad \Bigg| \qquad \int uv' = uv - \int u'v$$

Ex. 1 $\displaystyle\int \ln x\,dx = \int \underbrace{\ln x}_{u} \cdot \underbrace{1}_{v'}\,dx = x\ln x - \int \underbrace{\frac{1}{x}\cdot x}_{1}\,dx = x\ln x - x + C$

$v = x$

Ex. 2 (gamma function) $\quad \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \quad (x > 0)$

$$\Gamma(x+1) = \int_0^\infty \underbrace{t^x}_{u}\underbrace{e^{-t}}_{v'}\,dt = -t^x e^{-t}\Big|_{t=0}^{\infty} - \int_0^\infty x t^{x-1}(-e^{-t})\,dt$$

$v = -e^{-t}$

$$= 0 + x\int_0^\infty t^{x-1}e^{-t}\,dt = x\Gamma(x)$$

$$\lim_{t\to\infty} \frac{t^x}{e^t} = 0$$

n a pos. int., $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots 1 = n!$

$x! = \Gamma(x+1)$

$$\left(-\tfrac{1}{2}\right)! = \Gamma(1/2) = \sqrt{\pi}$$

Ex. 3 (§7.5 #26) $\displaystyle\lim_{x\to 0} \frac{\tan 4x}{\tan 5x} \overset{L'H}{=} \lim_{x\to 0} \frac{4\sec^2 4x}{\underbrace{5\sec^2 5x}_{\neq 0}} = \frac{4}{5}$

Ex. 4 (§7.5 #30) $\displaystyle\lim_{x\to\pi/2} \left(x - \frac{\pi}{2}\right)\tan x$

$0\cdot\infty$

$$\frac{\left(x-\frac{\pi}{2}\right)}{\cot x}$$

$$\lim_{x\to\pi/2} \frac{\sec^2 x}{-\left(x-\frac{\pi}{2}\right)^{-2}}$$

$$\overset{L'H}{=} \lim_{x\to\pi/2} \frac{1}{\underbrace{-\csc^2 x}_{\neq 0}} = \lim_{x\to\pi/2} -\sin^2 x = -1$$

Ex. 5 (§7.5 #28) $\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x - \tan x}{x \tan x} \overset{L'H}{=} \lim_{x\to 0}\frac{1-\sec^2 x}{\underbrace{x\sec^2 x + \tan x}_{\neq 0 \text{ for } x \text{ near } 0,\ x\neq 0}}$

$\sec^2 x\,(2 + 2x\tan x) \approx 2$

$\frac{1}{\tan x} - \frac{1}{x}$ $\qquad \frac{x\cot x - 1}{x}$ $\qquad \frac{0\cdot\infty - 1}{0}$ $\qquad \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0}\frac{-2\sec x\,(\sec x\tan x)}{\sec^2 x + x\cdot 2\sec^2 x\tan x + \sec^2 x}$

$\sec' x = \sec x \tan x$

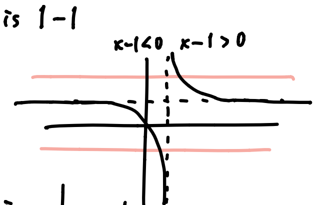
$= \lim_{x\to 0}\frac{-2\tan x}{\underbrace{2 + 2x\tan x}_{\neq 0}} = 0$

Ex. 6 (Chapter review #5) Show that $g(x) = \frac{x}{x-1}$ is equal to its inverse on the domain $\{x : x \neq +1\}$.

Can also show it has an inverse w/o explicitly writing one down

$g'(x) = \frac{(x-1) - x}{(x-1)^2} = \frac{-1}{(x-1)^2} < 0$

$\Rightarrow g$ is 1-1



g is also onto.

check if $g(g(x)) = x$ $\qquad y = \frac{x}{x-1}$

$\frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{\frac{x}{\cancel{x-1}}}{\frac{x}{\cancel{x-1}} - \frac{x-1}{\cancel{x-1}}} = \frac{x}{x-(x-1)} = x$

$g : A \to B$

$g^{-1} : g(A) \to A$

Ex. 7 (§7.5 #54) $\lim_{x\to\infty}\left(1 + \frac{r}{x}\right)^x$

$\lim_{x\to\infty} x\ln\left(1+\frac{r}{x}\right) = \lim_{x\to\infty}\frac{\ln(1 + r/x)}{1/x} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{1+r/x}\left(-r\cancel{x^{-2}}\right)}{-\cancel{x^{-2}}} = r\lim_{x\to\infty}\frac{1}{1+r/x} = r$

$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{r}{x}\right)} = e^{\lim_{x\to\infty} x\ln\left(1+\frac{r}{x}\right)} = e^r.$

Ex. 8 (C.R. #79) $\int_{1/3}^{2/3}\frac{dx}{\sqrt{1-x^2}} = \arcsin x\Big|_{1/3}^{2/3}$ $\qquad \arcsin' x = \frac{1}{\sqrt{1-x^2}}$

Ex. 9 (C.R. #74) $\int\frac{\cos(\ln x)\,dx}{x} \underset{du = \frac{dx}{x}}{\overset{u = \ln x}{=}} \int \cos u\,du = \sin u + C$

$= \sin \ln x + C$

$f\circ g = \int (f\circ g)' = \int f'(g)\,g'$

Or, $\cos = f'$, $g = \ln x$, $g' = \frac{1}{x}$

$\dashrightarrow \sin(\ln x) + C$

Ex. 10 (C.R. #85) $\int\frac{x\,dx}{\underbrace{\sqrt{1-x^4}}_{\sqrt{1-u^2}}} \underset{du = 2x\,dx}{\overset{u = x^2}{=}} \int\frac{\frac{1}{2}\,du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin(x^2) + C$

Ex. 11 (C.R. #122) Use L'Hôpital's rule to show,

$$\lim_{n\to\infty}\left(\frac{a^{1/n}+b^{1/n}}{2}\right)^n = \sqrt{ab} \qquad (a, b > 0)$$

$1^\infty \dashrightarrow \infty \cdot 0 \dashrightarrow \frac{0}{0}$

$$\lim_{x\to\infty} x\ln\left(\frac{a^{1/x}+b^{1/x}}{2}\right) = \lim_{x\to\infty}\frac{\ln\left(\frac{a^{1/x}+b^{1/x}}{2}\right)}{1/x} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{\cancel{2}}{a^{1/x}+b^{1/x}}\left(\frac{a^{1/x}\ln a\left(\cancel{-\frac{1}{x^2}}\right)+b^{1/x}\ln b\left(\cancel{-\frac{1}{x^2}}\right)}{\cancel{2}}\right)}{\cancel{-\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty}\frac{a^{1/x}\ln a + b^{1/x}\ln b}{a^{1/x}+b^{1/x}} = \frac{\ln a + \ln b}{2} \Rightarrow \lim_{n\to\infty}\left(\frac{a^{1/n}+b^{1/n}}{2}\right)^n = e^{\frac{\ln a + \ln b}{2}}$$

$$= e^{\ln a^{1/2}}\cdot e^{\ln b^{1/2}} = \sqrt{ab}$$

Ex. 12 (C.R. #88) $\int \sin\theta\cos\theta\, e^{\cos^2\theta+1}\,d\theta$

$u = \cos^2\theta + 1$

$du = 2\cos\theta(-\sin\theta)\,d\theta$

$$= \int -\frac{1}{2}e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{\cos^2\theta+1} + C$$

Ex. 13 (C.R. #41) $\int \underbrace{\frac{\arcsin x}{\sqrt{1-x^2}}}\,dx$

$(\arcsin x)(\arcsin' x) = \left(\frac{1}{2}(\arcsin x)^2\right)'$

$\frac{1}{2}\cdot 2\arcsin x\,\arcsin' x$

Or, $u = \arcsin x$

$du = \frac{dx}{\sqrt{1-x^2}}$

$$\to \int u\,du = \frac{1}{2}u^2 + C$$

$$= \frac{1}{2}(\arcsin x)^2 + C$$

Ex. 14 (C.R. #123) $$\lim_{x\to\infty}\frac{\int_2^x \frac{dt}{\ln t}}{\frac{x}{\ln x}} = 1$$

Ex. 15 Prove or give a counterexample: if $\lim_{x\searrow 0} f(x) = \lim_{x\searrow 0} g(x) = 0$, then $\lim_{x\searrow 0} f(x)^{g(x)} = 1$

$$f(x) = e^{-1/x},\quad g(x) = x$$

$$f(x)^{g(x)} = (e^{-1/x})^x = e^{-1} \not\to 1$$

disc. 4 (1F)

Today: • some examples of integration by parts

• review problems from the book

Integration by parts

$$u(b)v(b) - u(a)v(a) = \int_a^b (uv)'(t)\,dt = \int_a^b u'(t)v(t) + u(t)v'(t)\,dt$$

$$\Rightarrow \int_a^b \underbrace{u(t)v'(t)\,dt}_{u\,dv} = uv\Big|_a^b - \int_a^b \underbrace{v(t)u'(t)\,dt}_{v\,du}$$

$$\int uv' = uv - \int vu'$$

Ex. 1 $\int \ln x\,dx = \int \underbrace{\ln x}_{u} \cdot \underbrace{1}_{v'}\,dx = x\ln x - \int \frac{1}{x} x\,dx$

$v = x$

$= x\ln x - x + C$

Ex. 2 (gamma function) $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \quad (x>0)$

$\int_0^\infty = \lim_{R\to\infty}\int_0^R$

$$\Gamma(x+1) = \int_0^\infty \underbrace{t^x}_{u}\underbrace{e^{-t}}_{v'}\,dt = t^x(-e^{-t})\Big|_{t=0}^{t=\infty} - \int_0^\infty xt^{x-1}(-e^{-t})\,dt$$

$v = -e^{-t}$

$$= 0 + x\boxed{\int_0^\infty t^{x-1}e^{-t}\,dt} = x\Gamma(x)$$

Say n is a positive integer:

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2) = \cdots = n(n-1)\cdots 1 = n!$$

$$\left(-\tfrac{1}{2}\right)! = \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Ex. 3 (§7.5 #26) $\displaystyle \lim_{x\to 0}\frac{\tan 4x}{\tan 5x} \overset{L'H}{=} \lim_{x\to 0}\frac{4\sec^2 4x}{5\underbrace{\sec^2 5x}_{\approx 1 \text{ near } 0}} = \frac{4}{5}$

Ex. 4 (§7.5 #30) $\displaystyle \lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\tan x = \lim_{x\to\pi/2}\frac{\tan x}{1/(x-\frac{\pi}{2})} \quad \lim_{x\to\pi/2}\frac{\sec^2 x}{-\frac{1}{(x-\pi/2)^2}}$

$0\cdot\infty$

$\frac{0}{0} \quad \frac{\infty}{\infty}$

$$= \lim_{x\to\pi/2}\frac{x-\pi/2}{\cot x} \overset{L'H}{=} \lim_{x\to\pi/2}\frac{1}{\underbrace{-\csc^2 x}_{\approx 1 \text{ near } 0}} = -1$$

Ex. 5 (§7.5 #28) $\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x - \tan x}{x \tan x}$

$\frac{1}{\tan x} - \frac{1}{x}$

$\sec' = \sec \cdot \tan$

$$\overset{L'H}{=} \lim_{x\to 0}\frac{1-\sec^2 x}{\underbrace{\tan x + x\sec^2 x}_{\neq 0 \text{ near } 0}}$$

$$\overset{L'H}{=} \lim_{x\to 0}\frac{-2\sec x(\sec x\tan x)}{\underbrace{\sec^2 x + (\sec^2 x + x\cdot 2\sec^2 x\tan x)}_{\approx 2 \text{ near } 0}}$$

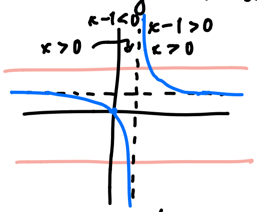
$$= \lim_{x\to 0}\frac{-2\tan x}{2+2x\tan x} = 0$$

Ex. 6 (Chapter review #5) Show that $g(x) = \frac{x}{x-1}$ is equal to its inverse on the domain $\{x : x \neq +1\}$.

Can also show it has an inverse w/o explicitly writing one down

$$g'(x) = \frac{(x-1)\cdot 1 - x\cdot 1}{(x-1)^2} = \frac{-1}{\underbrace{(x-1)^2}_{\geq 0}} < 0$$

$\Rightarrow g$ str. decreasing $\Rightarrow$ one-to-one



$g(g(x)) = x$

$$g(g(x)) = \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{\frac{x}{\cancel{x-1}}}{\frac{x}{\cancel{x-1}} - \frac{x-1}{\cancel{x-1}}} = \frac{x}{x-(x-1)} = x$$

Ex. 7 (§7.5 #54) $\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x$

$\frac{0}{0}$ $\frac{\infty}{\infty}$ $\frac{2}{\infty}$

$0\cdot\infty$

1^∞

Take a log. $\lim_{x\to\infty} x\ln\left(1+\frac{r}{x}\right) = \lim_{x\to\infty}\frac{\ln(1+r/x)}{1/x} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{1+r/x}\cdot -\frac{r}{\cancel{x^2}}}{-\frac{1}{\cancel{x^2}}}$

$$= \lim_{x\to\infty}\frac{r}{1+r/x} = r$$

Then

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x = \lim_{x\to\infty} e^{x\ln(1+r/x)} = e^{\lim_{x\to\infty} x\ln(1+r/x)} = e^r.$$

Ex. 8 (C.R. #79) $\int_{1/3}^{2/3}\frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{1/3}^{2/3}$

$(f\circ g)'(x) = f'(g(x))\,g'(x)$

Ex. 9 (C.R. #74) $\int\frac{\cos(\ln x)\,dx}{x} \underset{du = \frac{dx}{x}}{\overset{u=\ln x}{=}} \int \cos u\,du = \sin u + C = \sin\ln x + C$

$f' = \cos,\ g = \ln \longrightarrow \sin\ln x + C$

$f'(g)g' = (\cos\ln x)\frac{1}{x}$

Ex. 10 (C.R. #85) $\int\frac{x\,dx}{\sqrt{1-x^4}} \underset{du=2x\,dx}{\overset{u=x^2}{=}} \int\frac{\frac{1}{2}du}{\sqrt{1-u^2}} = \frac{1}{2}\int\frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin u + C$

$$= \frac{1}{2}\arcsin x^2 + C$$

Ex. 11 (C.R. #122) Use L'Hôpital's rule to show,

$$\lim_{n\to\infty}\left(\frac{a^{1/n}+b^{1/n}}{2}\right)^n = \sqrt{ab} \qquad (a, b > 0)$$

$$\ln\left(\lim_{x\to\infty}\left(\frac{a^{1/x}+b^{1/x}}{2}\right)^x\right) = \lim_{x\to\infty} x \ln \frac{a^{1/x}+b^{1/x}}{2} = \lim_{x\to\infty} \frac{\ln \frac{a^{1/x}+b^{1/x}}{2}}{1/x}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{\cancel{2}}{a^{1/x}+b^{1/x}}\cdot\frac{1}{\cancel{2}}\left(a^{1/x}\ln a\left(-\cancel{\frac{1}{x^2}}\right)+b^{1/x}\ln b\left(-\cancel{\frac{1}{x^2}}\right)\right)}{-\cancel{\frac{1}{x^2}}} = \lim_{x\to\infty}\frac{a^{1/x}\ln a + b^{1/x}\ln b}{a^{1/x}+b^{1/x}}$$

$$= \frac{\ln a + \ln b}{2}$$

orig. limit $= e^{\frac{1}{2}\ln a + \frac{1}{2}\ln b} = \sqrt{ab}$

Ex. 12 (C.R. #88)

$$\int \sin\theta\cos\theta\, e^{\cos^2\theta+1}\,d\theta = \int e^u\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}e^u + C$$

$$u = 1 + \cos^2\theta$$
$$du = 2\cos\theta\,(-\sin\theta)\,d\theta$$

$$= -\tfrac{1}{2}e^{\cos^2\theta+1} + C$$

Ex. 13 (C.R. #91)

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \tfrac{1}{2}(\arcsin x)^2$$

$$(\arcsin x)(\arcsin x)'$$
$$= \left(\tfrac{1}{2}(\arcsin x)^2\right)'$$

Ex. 14 (C.R. #123)

$$\lim_{x\to\infty}\frac{\int_2^x \frac{dt}{\ln t}}{\frac{x}{\ln x}}$$

Ex. 15 Prove or give a counterexample: if $\lim_{x\searrow 0} f(x) = \lim_{x\searrow 0} g(x) = 0$, then $\lim_{x\searrow 0} f(x)^{g(x)} = 1$

$x\searrow 0 \quad x\to 0^+$

$$\underset{\to 0}{f(x) = e^{-1/x}}, \quad \underset{\to 0}{g(x) = x}$$

$$f(x)^{g(x)} = (e^{-1/x})^x = e^{-1} \not\to 1$$

disc. 5 (1E)

More integrals

44. $\displaystyle\int \sin\sqrt{x}\,dx \;\overset{u=\sqrt{x}}{\underset{\substack{du=\frac{dx}{2\sqrt{x}}\\ 2u\,du=dx}}{=}}\; 2\int \underbrace{u}_{f}\,\underbrace{\sin u}_{g'}\,du = 2\left(-u\cos u - \int(-\cos u)\right) = 2(-u\cos u + \sin u) + C$

$= 2(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}) + C$

46. $\displaystyle\int \frac{\tan\sqrt{x}}{\sqrt{x}}\,dx \;\overset{u=\sqrt{x}}{\underset{\substack{du=\frac{dx}{2\sqrt{x}}\\ 2\,du=\frac{dx}{\sqrt{x}}}}{=}}\; \int 2\tan u\,du = 2\ln|\sec u| = 2\ln|\sec\sqrt{x}| + C$

35. $\displaystyle\int e^{\sqrt{x}}\,dx \;\overset{u=\sqrt{x}}{\underset{du=\frac{dx}{2\sqrt{x}}}{=}}\; 2\int \underbrace{e^{u}}_{g'}\,\underbrace{u}_{f}\,du = 2\left(ue^{u} - \int e^{u}\,du\right)$

$= 2(ue^{u} - e^{u}) + C = 2(\sqrt{x} - 1)e^{\sqrt{x}} + C$

45. $\displaystyle\int \sqrt{x}\,e^{\sqrt{x}}\,dx \;\leadsto\; \cdots\int u^2 e^{u}\,du$

81. $\displaystyle\int \arcsin^2 x\,dx$

$\displaystyle\int \underbrace{\arcsin^2 x}_{f}\cdot\underbrace{1}_{g'}\,dx = x\arcsin^2 x - 2\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx$

$u = \arcsin x \Rightarrow x = \sin u$

$du = \dfrac{dx}{\sqrt{1-x^2}}$

$\displaystyle\int u\sin u\,du = \sin u - u\cos u$

$= x\arcsin^2 x - 2\sin\arcsin x + \arcsin x\cos\arcsin x + C$

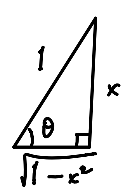
$= x\arcsin^2 x + \sqrt{1-x^2}\arcsin x - 2x + C$

Recall:

$\displaystyle\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + C$

$\cos\arcsin x = \sqrt{1-x^2}$

$x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2x + C$



$\arcsin x = \theta$

$\cos\arcsin x$

$= \cos\theta = \sqrt{1-x^2}$

Trigonometric integrals

$$\int \sin^m x \cos^n x \, dx$$

Reduction formulas

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \qquad (n \geq 2)$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \qquad (n \geq 2)$$

Ex. cos reduction formula

$$\int \underbrace{\cos^{n-1} x}_{u} \underbrace{\cos x}_{v'} \, dx = \cos^{n-1} x \sin x - \int \left((n-1) \cos^{n-2} x (-\sin x) \right) \sin x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, (1 - \cos^2 x) \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$\Rightarrow \int \cos^n x \, dx + (n-1) \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$$

$$n \int \cos^n x \, dx = \text{''}$$

$$\Rightarrow \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

e.g. $\int \cos^2 x \, dx = \frac{1}{2} \sin x \cos x + \frac{1}{2} x + C$

So, we can compute $\int \cos^n x \, dx$, $\int \sin^n x \, dx$ for any $n \geq 0$.

$$9 \to 7 \to 5 \to 3 \to 1$$
$$8 \to 6 \to 4 \to 2 \to 0$$

$$\int \sin^m x \cos^n x \, dx$$

$\boxed{m \text{ odd}}$

Split off one $\sin x$, rewrite $\sin^{m-1} x$ in terms of $\cos x$ & set $u = \cos x$

n odd

Same thing but split off $\cos x$ instead

$\boxed{m, n \text{ even}}$

Use $\cos^2 + \sin^2 = 1$ to write in terms of powers of sin or cos.

$$\int \cos^n x \sin^{m-1} x \sin x \, dx = \int \cos^n x (1 - \cos^2 x)^{\frac{m-1}{2}} \sin x \, dx \underset{du = -\sin x \, dx}{\overset{u = \cos x}{=}} -\int u^n (1 - u^2)^{\frac{m-1}{2}} \, du$$

$$\int \cos^n x \sin^m x \, dx = \int \cos^n x \left(1 - \cos^2 x\right)^{m/2} dx$$

Exs.

(i)
$$\int \cos^3 x \sin^5 x \, dx = \int \cos^3 x \sin^4 x \sin x \, dx = \int \cos^3 x \left(1 - \cos^2 x\right)^2 \sin x \, dx$$
$$\underset{du = -\sin x\,dx}{\overset{u = \cos x}{=}} \int u^3 (1-u^2)^2 (-du) = -\int u^3 (1 - 2u^2 + u^4) \, du$$
$$= -\int u^3 - 2u^5 + u^7 \, du = -\left(\frac{u^4}{4} - \frac{2u^6}{6} + \frac{u^8}{8}\right) + C$$
$$= -\frac{1}{4}\cos^4 x + \frac{1}{3}\cos^6 x - \frac{1}{8}\cos^8 x + C$$

(ii)
$$\int \cos^{56} x \sin^{62} x \, dx = \int \cos^{56} x \left(1 - \cos^2 x\right)^{31} dx$$

—

Trig. substitution $(a > 0)$

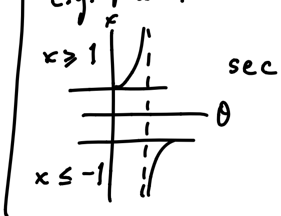
$\sqrt{a^2 - x^2} \dashrightarrow x = a \sin\theta \quad (|\theta| \le \pi/2)$

$\sqrt{x^2 + a^2} \dashrightarrow x = a \tan\theta \quad (|\theta| \le \pi/2)$

$\sqrt{x^2 - a^2} \dashrightarrow x = a \sec\theta \quad x \ge a,\ 0 \le \theta < \pi/2$
$\quad x \le -a,\ \pi \le \theta < \frac{3\pi}{2}$

$$\int \frac{1}{\sqrt{1 - x^2}} dx = \arcsin x + C$$
$x = \sin\theta \quad dx = \cos\theta \, d\theta \longrightarrow \theta = \arcsin x$
$$\int \frac{1}{\sqrt{1 - \sin^2\theta}} \cos\theta \, d\theta = \int d\theta = \theta + C$$

e.g. if a = 1:



$$\int \frac{dx}{1 - x^2} \underset{dx = \cos\theta\,d\theta}{\overset{x = \sin\theta}{=}} \int \frac{\cos\theta \, d\theta}{\cos^2\theta} = \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$
$$= \ln|\sec \arcsin x + \tan \arcsin x| + C \ \ldots$$

38.
$$\int \frac{dx}{\sqrt{12x - x^2}} = \int \frac{dx}{\sqrt{36 - (x-6)^2}} \overset{u = x-6}{=} \int \frac{du}{\sqrt{36 - u^2}} \underset{du = 6\cos\theta\,d\theta}{\overset{u = 6\sin\theta}{=}} \int \frac{\cos\theta \, d\theta}{\sqrt{36 - 36\sin^2\theta}} = \int \frac{\cos\theta}{6\cos\theta} d\theta$$
$$= \frac{1}{6}\int d\theta = \frac{1}{6}\theta + C = \frac{1}{6}\arcsin\left(\frac{x}{6} - 1\right) + C$$

$$\theta = \arcsin\frac{u}{6} = \arcsin\left(\frac{x}{6} - 1\right)$$

$$12x - x^2 = -(x - a)^2 + b$$
$$= -(x^2 - 2ax + a^2) + b$$
$$= -x^2 + 2ax + (\underbrace{b - a^2}_{0})$$
$$a = 6,\ b = a^2 = 36$$
$$= 36 - (x-6)^2$$

$$ax^2 + bx + c = a(x - c_0)^2 + c_1$$
$$u = x - c_0$$

disc. 5 (1F)

More integrals

44. $\int \sin\sqrt{x}\, dx \underset{\substack{du = \frac{dx}{2\sqrt{x}} \\ dx = 2\sqrt{x}\, du \\ = 2u\, du}}{\overset{u=\sqrt{x}}{=}} \int \sin u\, (2u\, du) = 2\underbrace{\int \underbrace{u}_{f}\, \underbrace{\sin u}_{g'}\, du}$

$= -u\cos u - \int 1\cdot(-\cos u)\, du$

$= \sin u - u\cos u$

46. $\int \frac{\tan\sqrt{x}}{\sqrt{x}}\, dx \underset{du = dx/2\sqrt{x}}{\overset{u=\sqrt{x}}{=}} \int (\tan u)\, 2\, du$

$= 2(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}) + C.$

$= 2\ln|\sec u| = 2\ln|\sec\sqrt{x}|.$

35. $\int e^{\sqrt{x}}\, dx \underset{du = dx/2\sqrt{x}}{\overset{u=\sqrt{x}}{=}} \int \underbrace{e^u}_{g'}\, \underbrace{2u}_{f}\, du$

$= 2ue^u - \int 2e^u$

$= 2(u-1)e^u + C = 2(\sqrt{x}-1)e^{\sqrt{x}} + C$

45. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx = \dots \int u^2 e^u\, du$

81. $\int \arcsin^2 x\, dx$

$\int \underbrace{\arcsin^2 x}_{f} \cdot \underbrace{1}_{g'}\, dx$

$= x\arcsin^2 x - 2\underbrace{\int \frac{x\arcsin x}{\sqrt{1-x^2}}\, dx}$

$u = \arcsin x$

$du = \frac{dx}{\sqrt{1-x^2}}$

$x = \sin u$

$\underbrace{\int u\sin u\, du}_{\sin u - u\cos u + C}$

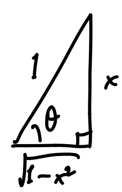
$= x\arcsin^2 x - 2\,\underline{\sin\arcsin}\, x + \arcsin x\, \underline{\cos\arcsin}\, x + C$

$= x\arcsin^2 x - 2x + \sqrt{1-x^2}\arcsin x + C$

Recall:

$\cos\arcsin x = \sqrt{1-x^2}$

$x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2x + C$



$\sin\theta = x$

$\arcsin x = \theta$

$\cos\arcsin x = \sqrt{1-x^2}$

Trigonometric integrals

$$\int \sin^m x \cos^n x \, dx$$

Reduction formulas

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \qquad (n \geq 2)$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \qquad (n \geq 2)$$

Ex. cos reduction formula

$$\int \underbrace{\cos^{n-1} x}_{f} \, \underbrace{\cos x}_{g'} \, dx = \cos^{n-1} x \sin x - \int \sin x \, (n-1) \cos^{n-2} x \, (-\sin x) \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$\underbrace{\int \cos^n x \, dx + (n-1) \int \cos^n x \, dx}_{n \int \cos^n x \, dx} = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$$

$$\Rightarrow \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

So, we can compute $\int \cos^n x \, dx$, $\int \sin^n x \, dx$ for any $n \geq 0$.

$$8 \to 6 \to 4 \to 2 \to 0$$
$$9 \to 7 \to 5 \to 3 \to 1$$

$$\int \cos^2 x \, dx = \frac{1}{2} \cos x \sin x + \frac{1}{2} x + C$$

$$\int \sin^m x \cos^n x \, dx$$

$m = 2k+1$

$\boxed{m \text{ odd}}$

Split off one $\sin x$, rewrite $\sin^{m-1} x$ in terms of $\cos x$ & set $u = \cos x$

n odd

Same thing but split off $\cos x$ instead

$\boxed{m, n \text{ even}}$

Use $\cos^2 + \sin^2 = 1$ to write in terms of powers of sin or cos.

$$\int \sin^{2k} x \cos^n x \sin x \, dx = \int (1 - \cos^2 x)^{\frac{m-1}{2}} \cos^n x \sin x \, dx \overset{\substack{u = \cos x \\ du = -\sin x \, dx}}{=} -\int (1-u)^{\frac{m-1}{2}} u^n \, du$$

$$\int \underbrace{\sin^m x}_{(\sin^2 x)^{m/2}} \cos^n x \, dx = \int (1-\cos^2 x)^{m/2} \cos^n x \, dx \leftarrow$$

Exs.

(i) $$\int \sin^5 x \cos^n x \, dx = \int \sin^4 x \cos^n x \sin x \, dx = \int (1-\cos^2 x)^2 \cos^n x \sin x \, dx$$

$$\underset{du = -\sin x\, dx}{\overset{u=\cos x}{=}} -\int (1-u^2)^2 u^n \, du = -\int (1 - 2u^2 + u^4) u^n \, du$$

$$= -\int u^n \, du + 2\int u^{n+2} \, du - \int u^{n+4} \, du$$

$$= -\frac{(\cos x)^{n+1}}{n+1} + 2\frac{(\cos x)^{n+3}}{n+3} - \frac{(\cos x)^{n+5}}{n+5} + C$$

(ii) $$\int \sin^6 x \cos^{70} x \, dx = \int (1-\cos^2 x)^3 \cos^{70} x \, dx$$

Trig. substitution $(a > 0)$

$\cos^2 + \sin^2 = 1$

$1 + \tan^2 = \sec^2$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta \, d\theta}{\cos\theta} = \arcsin x + C$$

$x = \sin\theta$ $\quad |\theta| \le \pi/2$

$dx = \cos\theta \, d\theta$

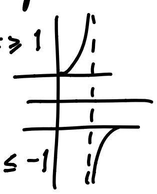
$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$\sqrt{a^2-x^2} \dashrightarrow x = a\sin\theta \quad (|\theta| \le \pi/2)$

$\sqrt{x^2+a^2} \dashrightarrow x = a\tan\theta \quad (|\theta| \le \pi/2)$

$\sqrt{x^2-a^2} \dashrightarrow x = a\sec\theta \quad x \ge a,\ 0 \le \theta < \pi/2$; $\quad x \le -a,\ \pi \le \theta < \frac{3\pi}{2}$

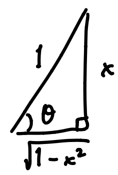
e.g. if $a = 1$:



$$\int \frac{dx}{1-x^2} \underset{dx = \cos\theta\, d\theta}{\overset{x = \sin\theta}{=}} \int \frac{\cos\theta \, d\theta}{\cos^2\theta} = \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta|$$

$$= \ln\Big|\underbrace{\sec \arcsin x}_{\frac{1}{\sqrt{1-x^2}}} + \underbrace{\tan \arcsin x}_{\frac{x}{\sqrt{1-x^2}}}\Big| + C$$

$$= \ln \frac{|x+1|}{\sqrt{1-x^2}} + C$$



$\sin\theta = x$

$\arcsin x = \theta$

$$\int \frac{dx}{(1-x^2)^{3/2}} \underset{dx = \cos\theta\, d\theta}{\overset{x = \sin\theta}{=}} \int \frac{\cos\theta \, d\theta}{(\cos^2\theta)^{3/2}} = \int \frac{d\theta}{\cos^2\theta} = \int \sec^2\theta \, d\theta = \tan\theta + C$$

$$= \tan \arcsin x + C = \frac{x}{\sqrt{1-x^2}} + C$$

38. $$\int \frac{dx}{\sqrt{12x - x^2}}$$

Complete the square: $12x - x^2 = -(x-a)^2 + b$

$= -(x^2 - 2ax + a^2) + b$

$= -x^2 + 2ax + (b - a^2)$

$\Rightarrow \quad a = 6, \quad b = a^2 = 36$

$$12x - x^2 = 36 - (x-6)^2$$

$$\int \frac{dx}{\sqrt{36-(x-6)^2}} \overset{u=x-6}{\underset{du=dx}{=}} \int \frac{du}{\sqrt{36-u^2}} \overset{u = 6\sin\theta}{\underset{du = 6\cos\theta\, d\theta}{=}} \int \frac{6\cos\theta\, d\theta}{\cos\theta} = 6\theta + C$$

$$\theta = \arcsin\left(\frac{u}{6}\right) = \arcsin\left(\frac{x}{6} - 1\right)$$

$$= 6 \arcsin\left(\frac{x}{6} - 1\right) + C$$

$\sqrt{7-u^2}$

$u = \sqrt{7} \sin\theta$

disc. 6 (1E)

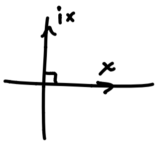
cosh & sinh

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh = \frac{\sinh}{\cosh}$$

$$\operatorname{sech} = \frac{1}{\cosh} \quad \text{etc.}$$

vs cos, sin $\quad (e^{ix} = \cos x + i \sin x, \ x \in \mathbb{R})$



$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Pythagorean identity (hyperb. version)

$$\cosh^2 x - \sinh^2 x$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left(\cancel{e^{2x}} + \cancel{e^{-2x}} + 2\cancel{e^x e^{-x}} - \cancel{e^{2x}} - \cancel{e^{-2x}} + 2\cancel{e^x e^{-x}}\right)$$

$$= 1$$

$$\frac{d}{dx}\cosh x = \frac{e^x - e^{-x}}{2}$$

$$= \sinh x$$

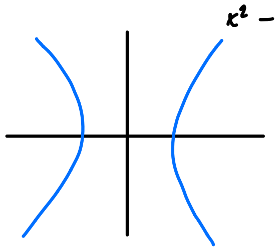
$$\frac{d}{dx}\sinh x = \frac{e^x - - e^{-x}}{2}$$

$$= \cosh x$$

$$\sin \xrightarrow{d/dx} \cos \to -\sin \to -\cos \to \sin \to \dots$$

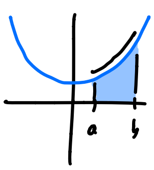
Geometric interpretation $\quad x = \cosh t, \ y = \sinh t$

$x^2 - y^2 = 1$



$(\pm \cosh t, \sinh t)$

Another note: arc length



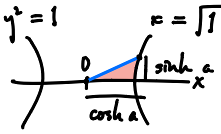
$f(t) = \cosh t \qquad \cosh t = \sqrt{1 + \sinh^2 t}$

$$\text{Arc length from } a \text{ to } b = \int_a^b \sqrt{1 + (f'(t))^2}\, dt = \int_a^b \sqrt{1 + \sinh^2 t}\, dt = \int_a^b \cosh t\, dt$$

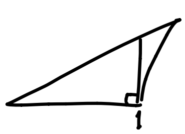
An area computation

$x = \frac{\cosh a}{\sinh a} y$

$x^2 - y^2 = 1 \qquad x = \sqrt{1 + y^2}$



$$\int_0^{\sinh a} \sqrt{1 + y^2} - \frac{\cosh a}{\sinh a} y \, dy$$



$$= \underbrace{\int_0^{\sinh a} \sqrt{1 + y^2}\, dy} - \frac{\cosh a}{\cancel{\sinh a}} \cdot \frac{1}{2} \sinh^{\cancel{2}} a$$

rec: $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 = \dots$$

$$y = \sinh u, \quad dy = \cosh u \, du$$

$$\sqrt{1+y^2} = \sqrt{1+\sinh^2 u} = \cosh u$$

$$\int_0^a \cosh^2 u \, du$$

$$= \frac{1}{2}\int_0^a \cosh 2u + 1 \, du = \frac{a}{2} + \frac{1}{2}\int_0^a \cosh 2u \, du$$

note $\left(\frac{1}{2}\sinh 2u\right)' = \frac{1}{2}\cdot 2\cosh 2u$

$$= \frac{a}{2} + \frac{1}{4}\sinh 2u \Big|_0^a = \frac{a}{2} + \frac{1}{4}\sinh 2a$$

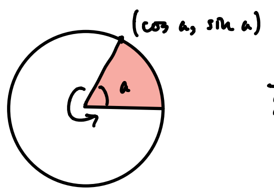
$$\Rightarrow \quad \text{area} = \frac{a}{2} + \frac{1}{4}\sinh 2a - \underbrace{\frac{1}{2}\cosh a \sinh a}$$

$$\frac{1}{2}\left(\frac{e^a + e^{-a}}{2}\right)\left(\frac{e^a - e^{-a}}{2}\right) = \frac{1}{8}\left(e^{2a} + \cancel{e^{-a}e^{a}} - \cancel{e^{-a}e^{a}} - e^{-2a}\right) = \frac{1}{4}\sinh 2a$$

$$= a/2, \text{ as desired.}$$

$$2\sinh x \cosh x = \sinh 2x$$

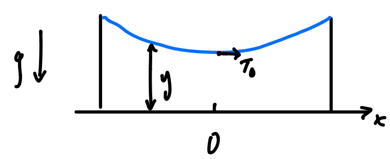
Compare:



$$\frac{a}{2\pi}\cdot\pi = a/2$$

Catenary equation

$$y'' = \frac{\rho g A}{T_0}\sqrt{1+(y')^2}$$



ρ = density of the rope
g = accel. due to gravity
A = cross-sectional area of rope

Set $a = T_0/\rho g A$, $y(x) = a\cosh x/a$, so $y'(x) = \sinh x/a$, $y''(x) = 1/a \cosh x/a$

$$\frac{1}{a}\sqrt{1+(\sinh x/a)^2} = \frac{1}{a}\cosh x/a = y''(x)$$

Ex.

32. $\operatorname{artanh} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| \quad (|x| \le 1)$

$y = \sinh x = \frac{e^x + e^{-x}}{2}$

$y + \sqrt{y^2+1} = \sinh x + \cosh x$

$= \frac{e^x - \cancel{e^{-x}}}{2} + \frac{e^x + \cancel{e^{-x}}}{2} = e^x$

$\operatorname{arsinh} y = \ln(y + \sqrt{y^2+1})$

$\frac{d}{dx}\operatorname{artanh} x = \frac{1}{1-x^2}$

Take derivative of both sides:

$\frac{1}{1-x^2} \qquad \frac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{(1-x)\cdot 1 - (1+x)\cdot(-1)}{(1-x)^2} = \frac{1}{\cancel{2}}\cdot\frac{\cancel{1-x}}{1+x}\cdot\frac{\cancel{2}}{(1-x)^{\cancel{2}}} = \frac{1}{1-x^2}$

Thus by integration, $\operatorname{artanh} x - \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| \equiv C$, some const. C.

Plug in $x=0$: $\operatorname{artanh} x - \frac{1}{2}\ln\left|\frac{1+0}{1-0}\right| = 0 \Rightarrow C = 0$

More hyperbolic substitution

$\sqrt{x^2+a^2} \dashrightarrow x = a\sinh u,\ dx = a\cosh u\,du,\ \sqrt{x^2+a^2} = a\cosh u$

$\sqrt{x^2-a^2} \dashrightarrow x = a\cosh u,\ dx = a\sinh u\,du,\ \sqrt{x^2-a^2} = a\sinh u$

21. $\int\sqrt{x^2-2}\,dx = \int\sqrt{2}\sinh u\sqrt{2}\sinh u\,du = 2\int\sinh^2 u\,du$

$x = \sqrt{2}\cosh u$

$dx = \sqrt{2}\sinh u\,du$

$= \int\cosh 2u - 1\,du$

$\sinh^2 u = \left(\frac{e^u - e^{-u}}{2}\right)^2$

$= \frac{1}{4}(e^{2u} + e^{-2u} - 2\cancel{e^u e^{-u}})$

$= \frac{1}{2}(\cosh 2u - 1)$

$\sqrt{2\cosh^2 u - 2} = \sqrt{2}\sqrt{\cosh^2 u - 1} = \sqrt{2}\sinh u$

$(u > 0)$

$u = \operatorname{arcosh}\left(\frac{x}{\sqrt{2}}\right)$

$-u + \int\cosh 2u$

$-u + \frac{1}{2}\sinh 2u + C$

$= -u + \frac{1}{\cancel{2}}\cdot\cancel{2}\sinh u\cosh u$

$= -u + \cosh u\sqrt{\cosh^2 u - 1}$

$= -\operatorname{arcosh}\frac{x}{\sqrt{2}} + \frac{x}{\sqrt{2}}\sqrt{x^2/2 - 1}$

$-\operatorname{arcosh}\frac{x}{\sqrt{2}} + \frac{1}{2}\sinh\left(2\operatorname{arcosh}\frac{x}{\sqrt{2}}\right) + C$

37. $\int\frac{\operatorname{artanh} x}{x^2-1}\,dx$

$u = \operatorname{artanh} x$

$du = \frac{1}{1-x^2}\,dx$

$\int u(-du) = -\frac{1}{2}u^2 + C$

$= -\frac{1}{2}\operatorname{artanh}^2 x + C$

incidentally,

$\operatorname{arcosh} z = \ln(z + \sqrt{z+1}\sqrt{z-1})$

$\sqrt{(z+1)(z-1)}$

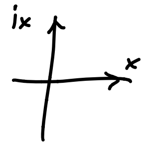
$= \sqrt{z^2-1}$

disc. 6 (1F)

cosh & sinh

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

vs cos, sin (recall: $e^{ix} = \cos x + i \sin x$, $x \in \mathbb{R}$)



$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

—

Pythagorean identity (hyperb. version)

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left(\cancel{e^{2x}} + \cancel{e^{-2x}} + 2e^x e^{-x} - \cancel{e^{2x}} - \cancel{e^{-2x}} + 2e^x e^{-x}\right) = 1$$

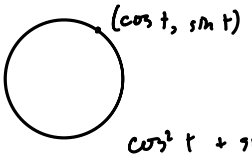
—

$$\frac{d}{dx} \cosh x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$$

—

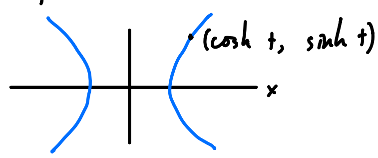
Geometric interpretation

recall:



$\cos^2 t + \sin^2 t = 1$

for the hyperbolic functions:



$x^2 - y^2 = 1$

$\underline{\cosh^2 - \sinh^2 = 1}$

Another note: arc length

recall: arc length of f from a to b is $\int_a^b \sqrt{1 + (f')^2}\, dx$

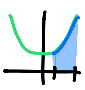
Note that

$\cosh^2 x = 1 + \sinh^2 x$

$\cosh x = \sqrt{1 + \sinh^2 x}$

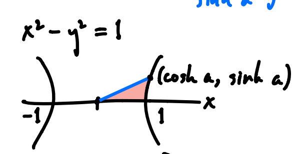
so $\sqrt{1 + (\cosh' x)^2} = \cosh x$, hence

$$\int_a^b \cosh x \, dx = \int_a^b \sqrt{1 + (\cosh' x)^2}\, dx,$$ i.e. the arc length is the same as the area under the curve.



An area computation

$x = \frac{\cosh a}{\sinh a} y$



$x = +\sqrt{1 + y^2}$ ∫

$$\int_0^{\sinh a} \sqrt{1 + y^2} - \frac{\cosh a}{\sinh a} y \, dy$$

$$= \underbrace{\int_0^{\sinh a} \sqrt{1 + y^2}\, dy}_{1} - \frac{\cosh a}{\cancel{\sinh a}} \cdot \frac{(\sinh a)^{\cancel{2}}}{2}$$

rec: $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$

$\cosh x = \sqrt{1 + \sinh^2 x}$

$$\downarrow$$

$$y = \sinh x$$
$$dy = \cosh x \, dx \qquad\qquad \sinh a = \sinh x$$

$$\int_0^a \cosh x (\cosh x \, dx) = \int_0^a \cosh^2 x \, dx$$

$$= \frac{1}{2}\int_0^a \cosh 2x + 1 \, dx = \underbrace{\frac{1}{2}\int_0^a \cosh 2x \, dx}_{\frac{1}{4}\sinh 2x \Big|_0^a} + \frac{1}{2}a$$

$$\left(\frac{1}{4}\sinh 2x\right)' = \frac{1}{2}\cosh 2x$$

$$= \frac{a}{2} + \underbrace{\frac{1}{4}\sinh 2a - \frac{1}{2}\sinh a \cosh a}_{=0}$$

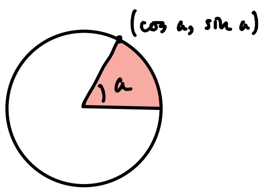
$$\sinh a \cosh a \to \left(\frac{e^a - e^{-a}}{2}\right)\left(\frac{e^a + e^{-a}}{2}\right)$$

$$= \frac{1}{4}\left(e^{2a} - \cancel{e^{-a}e^{a}} + \cancel{e^{a}e^{-a}} - e^{-2a}\right)$$

$$= \frac{1}{2}\left(\frac{e^{2a} - e^{-2a}}{2}\right) = \frac{1}{2}\sinh 2a$$

$$= a/2.$$

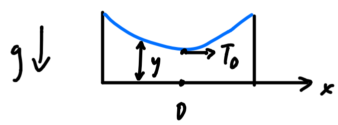
Compare:



$$\frac{a}{2\pi}\cdot\pi = a/2.$$

<u>Catenary equation</u>

$$y'' = \frac{\rho g A}{T_0}\sqrt{1 + (y')^2}$$



ρ = density of rope
g = accel. due to gravity
A = cross-sectional area

Set $a = T_0/\rho g A$, $y(x) = a\cosh x/a$, so that $y'(x) = \sinh x/a$, $y''(x) = \frac{1}{a}\cosh\frac{x}{a}$.

$$\text{RHS} = \frac{1}{a}\sqrt{1 + (\sinh x/a)^2} = \frac{1}{a}\cosh x/a = y''(x)$$

so this y satisfies the equation.

Ex. (if we know $(\operatorname{artanh} x)' = \frac{1}{1-x^2}$) $\tanh = \frac{\sinh}{\cosh}$ $\operatorname{sech} = \frac{1}{\cosh}$

32. $\operatorname{artanh} x = \underbrace{\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|}$ $(|x| \le 1)$

Take deriv. of both sides:

$$LHS' = \frac{1}{1-x^2}, \quad RHS' = \frac{1}{2}\left(\frac{1-x}{1+x}\right)\frac{(1-x)1-(1+x)(-1)}{(1-x)^2} = \frac{1}{\cancel{2}}\cdot\frac{\cancel{1-x}}{1+x}\cdot\frac{\cancel{2}}{(1-x)^{\cancel{2}}} = \frac{1}{1-x^2}$$

$$1 - \cancel{x} + (1+\cancel{x})$$

$$\Rightarrow \operatorname{artanh} x - \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| \equiv C$$

at $x=0$, $\operatorname{artanh} x - \frac{1}{2}\cdots = \operatorname{artanh} 0 - \frac{1}{2}\ln\left|\frac{1+0}{1-0}\right| = 0 \Rightarrow C = 0$

Ex. $y = \sinh x$

$$y + \sqrt{1+y^2} = \sinh x + \cosh x = \frac{e^x - \cancel{e^{-x}}}{2} + \frac{e^x + \cancel{e^{-x}}}{2} = e^x$$

$$\Rightarrow x = \ln\left(y + \sqrt{1+y^2}\right)$$

i.e. $\operatorname{arsinh} y = \ln\left(y + \sqrt{1+y^2}\right)$

More hyperbolic substitution

$$\sqrt{x^2+a^2} \dashrightarrow x = a\sinh u, \quad dx = a\cosh u\,du, \quad \sqrt{x^2+a^2} = a\cosh u$$

$$\sqrt{x^2-a^2} \dashrightarrow x = a\cosh u, \quad dx = a\sinh u\,du, \quad \sqrt{x^2-a^2} = a\sinh u$$

21. $\int \sqrt{x^2-2}\,dx \underset{\substack{x=\sqrt{2}\cosh u \\ dx = \sqrt{2}\sinh u\,du}}{=} \int \sqrt{2}\sinh u\,\sqrt{2}\sinh u\,du = 2\int \sinh^2 u\,du$ $u = \operatorname{arcosh}\frac{x}{\sqrt{2}}$

$$= \int \cosh 2u - 1\,du$$

$$= -u + \frac{1}{2}\sinh 2u + C$$

$$= -u + \frac{1}{2}\sinh u\cosh u + C$$

$$= -u + \frac{1}{2}\sqrt{\cosh^2 u - 1}\,\cosh u + C$$

$$= -\operatorname{arcosh}\frac{x}{\sqrt{2}} + \frac{x}{2\sqrt{2}}\sqrt{\left(x/\sqrt{2}\right)^2 - 1} + C$$

$$\sinh^2 u = \left(\frac{e^u - e^{-u}}{2}\right)^2 = \frac{1}{4}\left(e^{2u} + e^{-2u} - 2\right) = \frac{1}{2}(\cosh 2u - 1)$$

$$\sqrt{(\sqrt{2}\cosh u)^2 - 2} = \sqrt{2(\cosh^2 u - 1)} = \sqrt{2}\sinh u$$

37. $\int \frac{\operatorname{artanh} x}{x^2-1}\,dx$

$$u = \operatorname{artanh} x$$

$$du = \frac{dx}{1-x^2}$$

disc. 7 (1E)

Partial fractions

26. $\displaystyle\int \frac{dx}{x^3-3x^2+4}$

$x=-1,\quad (-1)^3-3(-1)^2+4 = -1-3+4=0$

$x^3-3x^2+4 = (x+1)(x^2+ax+b) \longrightarrow = (x+1)(x^2-4x+4)$

$= x^3+ax^2+bx+x^2+ax+b \qquad = (x+1)(x-2)^2$

$-3=a+1$

$0=b+a \qquad \Rightarrow a=-4$

$4=b$

$$\frac{1}{x^3-3x^2+4} = \frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

1st method: $A + B\frac{x+1}{x-2}+C\frac{x+1}{(x-2)^2} = \frac{1}{(x-2)^2}$ at -1: $A=\frac{1}{9}$

$A\frac{(x-2)^2}{x+1}+B(x-2)+C = \frac{1}{x+1}$ at 2: $C=\frac{1}{3}$

$A\frac{x-2}{x+1}+B+\frac{C}{x-2} = \frac{1}{(x+1)(x-2)}$ at 1: $\frac{1}{9}\left(\frac{-1}{2}\right)+B-\frac{1}{3} = \frac{1}{2\cdot(-1)}$

$\Rightarrow B = -\frac{1}{2}+\frac{1}{3}+\frac{1}{18}$

$= -\frac{3}{36}+\frac{1}{3\cdot 6} = -\frac{1}{9}$

2nd method: clear all denominators

$A(x-2)^2+B(x-2)(x+1)+C(x+1) = 1$

$A(x^2-4x+4)+B(x^2-x-2)+C(x+1)-1=0$

$x^2(A+B)+(-4A-B+C)x+(4A-2B+C-1)=0$

$$\Rightarrow \begin{cases} A+B=0 \\ -4A-B+C=0 \\ 4A-2B+C-1=0 \end{cases}$$

$$\Rightarrow \int\frac{dx}{x^3-3x^2+4} = \int\frac{1/9}{x+1}dx + \int\frac{-1/9}{x-2}dx + \int\frac{1/3}{(x-2)^2}dx$$

$$= \frac{1}{9}\ln|x+1| - \frac{1}{9}\ln|x-2| - \frac{1}{3}(x-2)^{-1} + C$$

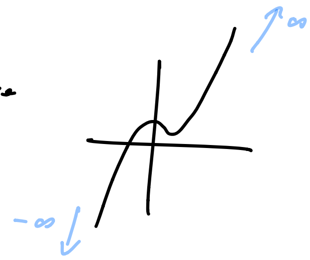
59. $\displaystyle\int\frac{dx}{(x-a)(x-b)} = \int\frac{1}{a-b}\left(\frac{1}{x-a}-\frac{1}{x-b}\right)dx = \frac{1}{a-b}\left(\ln|x-a|-\ln|x-b|+C\right)$

$(a\neq b)$

$$= \frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right| + C$$

What are the possible orders for irreducible polynomials over $\mathbb{R}$?

Note that every polynomial of <u>odd</u> order over $\mathbb{R}$ must have at least one root.



$x^n + \cdots$

This is by continuity + IVT

$\longrightarrow$ no irreducible polynomials over $\mathbb{R}$ of odd order ≥ 3

But in fact, more is true: if $\underset{\deg \geq 1}{\underline{P,}}$ is an irreducible polynomial over $\mathbb{R}$, then it has degree 1 or 2.

Suppose P has degree ≥ 3.

ζ 'zeta'
ξ 'xi'

<u>Thm.</u> (Fundamental thm. of algebra)

If P is a polynomial with complex coefficients & $\deg P \geq 1$, then P has a root over $\mathbb{C}$.

Fact: if P has a complex root ξ, then $\overline{\xi}$ is also a root. $(\overline{a+bi} = a - bi)$

By the thm., P has a complex root ξ. Thus, $\overline{\xi}$ is also a root.

So $P = \underbrace{(x-\xi)(x-\overline{\xi})}_{\text{is a real quadratic}} Q$ where Q is another polynomial. Continuing this reasoning, can factor Q over $\mathbb{R}$.

Improper integrals

$$\int_0^1 \ln x \, dx := \lim_{\varepsilon \searrow 0} \int_\varepsilon^1 \ln x \, dx = \lim_{\varepsilon \searrow 0} x \ln x - x \Big|_\varepsilon^1$$

$$= \lim_{\varepsilon \searrow 0} \left(1 \cancel{\ln 1} - 1 - (\varepsilon \ln \varepsilon - \varepsilon)\right) = \lim_{\varepsilon \searrow 0} (-1 - \varepsilon \ln \varepsilon + \varepsilon)$$

$$\lim_{\varepsilon \searrow 0} \varepsilon \ln \varepsilon = \lim_{\varepsilon \searrow 0} \frac{\ln \varepsilon}{1/\varepsilon} \overset{L'H}{=} \lim_{\varepsilon \searrow 0} \frac{1/\varepsilon}{-1/\varepsilon^2} = \lim_{\varepsilon \searrow 0} (-\varepsilon) = 0$$

$$= -1$$

$$\int_0^\infty \cdots = \lim_{b \to \infty} \int_0^b \cdots$$

Sequences

$(a_n) \qquad a_1, a_2, a_3, \ldots$

$\lim_{n\to\infty} a_n$?

Defn. of limit

We say $\lim_{n\to\infty} a_n = a$ if:

for all $\varepsilon > 0$, there exists N s.t. $n > N \Rightarrow |a_n - a| < \varepsilon$.

$$\frac{n+2}{n^2-6} = \frac{1/n + 2/n^2}{1 - 6/n^2}$$

$$\lim_{x\to\infty} \frac{x+2}{x^2-6}$$

Ex. $a_1 = 1, \quad a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right)$

Does this seq. converge? If so, what is the limit?

Claim: (i) $a_n^2 \geq 2$ for $n \geq 2$

(ii) $a_{n+1} \leq a_n$ for $n \geq 2$

induction: to see $P(n)$ is true for all $n \geq 1$, enough to show:
(i) $P(1)$ is true
(ii) $P(n) \Rightarrow P(n+1)$

$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \ldots$

By induction: $a_2 = \frac{3}{2} \Rightarrow a_2^2 \geq \frac{9}{4} \geq 2$

Now if $a_n^2 \geq 2$, we have

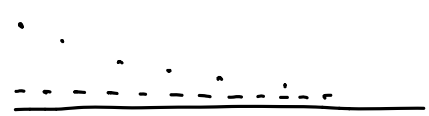
$$a_{n+1}^2 - 2 = \frac{1}{4}\left(a_n^2 + \frac{4}{a_n^2} + 4\right) - 2 = \left(\frac{a_n}{2}\right)^2 + \left(\frac{1}{a_n}\right)^2 - 1$$
$$= \left(\frac{a_n}{2} - \frac{1}{a_n}\right)^2 \geq 0$$

so (i) follows by induction.

For (ii):
$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{1}{2}a_n + \frac{1}{a_n} \leq a_n$$
iff. $\frac{1}{a_n} \leq \frac{a_n}{2}$, i.e. iff. $a_n^2 \geq 2$.

But we know this from (i).

So a_n is decreasing & bounded below, so $a_n \to x$ for some x.



What's the limit? If we know $a_n \to x$, then also $a_{n-1} \to x$.

So $x = \frac{1}{2}\left(x + \frac{2}{x}\right) \Rightarrow 2x = x + \frac{2}{x} \Rightarrow x = \frac{2}{x} \Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2} \Rightarrow x = \sqrt{2}$ since $a_n > 0$.

disc. 7 (1F)

Partial fractions

26. $\displaystyle\int \frac{dx}{\underbrace{x^3 - 3x^2 + 4}}$

$\downarrow$

$(x+1)(x^2 + ax + b) \longrightarrow (x+1)(x^2 - 4x + 4)$

$= (x+1)(x-2)^2$

$= x^3 + ax^2 + bx + x^2 + ax + b$

$= x^3 + (a+1)x^2 + (a+b)x + b$

$\left.\begin{array}{l} a+1 = -3 \\ a+b = 0 \end{array}\right\} \Rightarrow a = -4$

$b = 4$

$$\frac{1}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$-(x-2)^{-1}$

$0 = A(x-2)^2 + B(x-2)(x+1) + C(x+1) - 1$

$= A(x^2 - 4x + 4) + B(x^2 - x - 2) + C(x+1) - 1$

$= \underbrace{(A+B)}x^2 + \underbrace{(-4A - B + C)}x + \underbrace{(4A - 2B + C - 1)}$

$\Rightarrow \begin{cases} A + B = 0 \\ -4A - B + C = 0 \\ 4A - 2B + C - 1 = 0 \end{cases}$

$C = 4A + B$

$C = 1 - 4A + 2B$

$\Rightarrow 8A = 1 + 3B$

$B = -A$

$9A = 1 \Rightarrow A = 1/9$

$B = -1/9$

$C = 1/3$

$$\Rightarrow \int \frac{dx}{x^3 - 3x^2 + 4} = \frac{1}{9}\ln(x+1) - \frac{1}{9}\ln(x-2) - \frac{1}{3}(x-2)^{-1} + C$$

59. $\displaystyle\int \frac{dx}{(x-a)(x-b)} = \int \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right)dx = \frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right| + C$

$(a \neq b)$

$$\frac{1}{x-a} - \frac{1}{x-b} = \frac{x-b-x+a}{(x-a)(x-b)} = \frac{a-b}{(x-a)(x-b)}$$

What are the possible orders for irreducible polynomials over $\mathbb{R}$?

First, can see there are no irreducible polynomials of odd order ≥ 3 over $\mathbb{R}$.

$x^n + \cdots$, n odd

Has to go to ∞ on one side, $-\infty$ on other side

Thus by the IVT, the poly. must have a root.

<u>Thm.</u> If P is a nonconst. irreducible real polynomial, then P has order 1 or 2.

<u>Fact 1:</u> P has a root over $\mathbb{C}$ (by the Fundamental Thm. of Algebra)

<u>Fact 2:</u> If ξ is a root of P, then $\bar{\xi}$ is also a root. $(\overline{a+bi} = a - bi)$

<u>Pf.</u> Expand $P(x) = \underbrace{(x-a_1)(x-a_2)\cdots(x-a_k)}_{\text{linear}}\underbrace{(x-\xi_1)(x-\bar{\xi}_1)}_{\text{irr. quadratic}}\cdots\underbrace{(x-\xi_j)(x-\bar{\xi}_j)}$

where $a_i \in \mathbb{R}$, $\xi_j \in \mathbb{C}\setminus\mathbb{R}$

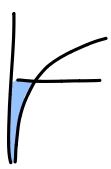
$$(x-\xi)(x-\bar{\xi}) = x^2 - (\xi+\bar{\xi})x + |\xi|^2$$

Improper integrals

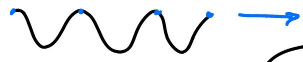
$$\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^3} = \lim_{b\to\infty} -\frac{1}{2}x^{-2}\Big|_1^b = \lim_{b\to\infty}\left(-\frac{1}{2}b^{-2} + \frac{1}{2}\right) = \frac{1}{2}$$

$$\int_0^1 \ln x\,dx = \lim_{\varepsilon\searrow 0}\int_\varepsilon^1 \ln x\,dx = \lim_{\varepsilon\searrow 0}(x\ln x - x)\Big|_\varepsilon^1 = \lim_{\varepsilon\searrow 0}(1\cancel{\ln 1}^{\,0} - 1 - \varepsilon\ln\varepsilon + \varepsilon)$$
$$= -1$$

$$\lim_{\varepsilon\searrow 0}\varepsilon\ln\varepsilon = \lim_{\varepsilon\searrow 0}\frac{\ln\varepsilon}{1/\varepsilon} \overset{\text{L'H}}{=} \lim_{\varepsilon\searrow 0}\frac{1/\varepsilon}{-1/\varepsilon^2} = \lim_{\varepsilon\searrow 0}(-\varepsilon) = 0$$



Sequences $\quad a(n) \quad f: \mathbb{Z}^+ \to \mathbb{R}$



$\boxed{\cos 2\pi x}$ if $x = n$, $\cos 2\pi n = (-1)^{2n} = 1$

$a_n: \quad a_1, a_2, a_3, a_4, a_5, \ldots$

$\lim_{x\to\infty} \frac{1}{x^2 + \sin x}$

$\lim_{n\to\infty} \frac{1}{n^2+\sin n}$

Defn. of limit We say $a_n \to a \in \mathbb{R}$ if

for every $\varepsilon > 0$, there exists N such that $n > N \Rightarrow |a_n - a| < \varepsilon$.

$a_n \to \infty$ if for every $M > 0$, " ________ $\Rightarrow a_n > M$

Ex. $a_1 = 1, \quad a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right) \qquad x = \frac{1}{2}\left(x + \frac{2}{x}\right)$

Does this seq. converge? If so, what is the limit?

Claim (i) $a_n^2 \geq 2 \quad (n \geq 2)$
(ii) $a_{n+1} \leq a_n \quad (n \geq 2)$

Note: induction

To prove a statement $P(n)$ for all n, it is sufficient to show
(i) $P(1)$ is true
(ii) $P(n) \Rightarrow P(n+1)$

$$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow P(4) \Rightarrow \ldots$$

Pf. of claim (i) $a_2 = \frac{3}{2}$ so true for $n=2$.

So assume $a_n^2 \geq 2$. Then

$$a_{n+1}^2 - 2 = \frac{1}{4}\left(a_n + \frac{2}{a_n}\right)^2 - 2$$
$$= \frac{1}{4}\left(a_n^2 + 4 + \frac{4}{a_n^2}\right) - 2 = \left(\frac{a_n}{2}\right)^2 + 1 + \left(\frac{1}{a_n}\right)^2 - 2$$
$$= \left(\frac{a_n}{2}\right)^2 + \left(\frac{1}{a_n}\right)^2 - 1 = \left(\frac{a_n}{2} - \frac{1}{a_n}\right)^2 \geq 0$$

i.e. $a_{n+1}^2 \geq 2$, so (i) follows by induction

(ii) $$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{a_n}{2} + \frac{1}{a_n} \leq a_n$$

iff. $\frac{1}{a_n} \leq \frac{a_n}{2}$ iff. $2 \leq a_n^2$

so this follows from (i).

Now use the following fact: if a_n is decreasing & bounded below, then the limit exists.

So $a_n \to x$ for some $x > 0$.

Hence $x = \frac{1}{2}\left(x + \frac{2}{x}\right) \Rightarrow 2x = x + \frac{2}{x} \Rightarrow x = \frac{2}{x} \Rightarrow x^2 = 2$

$\Rightarrow x = \sqrt{2}$.

disc. 8

$a_1, a_2, a_3, \ldots$

Series Consider a sequence $a_n : \mathbb{Z}^+ \to \mathbb{R}$. We define $s_N = a_1 + \cdots + a_N = \sum_{j=1}^{N} a_j$.

If $\lim_{N\to\infty} s_N = s$ exists, we write $\sum_{j=1}^{\infty} a_j = s$.

$$\sum_{j=1}^{\infty} 2^{-j} = 1, \quad \sum_{j=1}^{\infty} 1 = \infty, \quad \sum_{n=1}^{\infty} -\frac{1}{n} = \infty, \quad \sum_{n=1}^{\infty} (-1)^n \text{ DNE}$$

Geometric series

$$\sum_{n=1}^{\infty} r^n = \frac{1}{1-r} \quad \text{if } |r| < 1$$

$$S_N = 1 + r + r^2 + \cdots + r^N$$

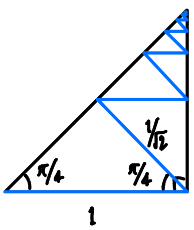
$$(1-r)S_N = (1 + r + \cdots + r^N) - r(1 + r + \cdots + r^N)$$
$$= 1 + \cancel{r} + \cancel{r^2} + \cdots + \cancel{r^N} - \cancel{r} - \cancel{r^2} - \cdots - r^{N+1} = 1 - r^{N+1}$$

$$\overset{r \neq 1}{\Rightarrow} \quad S_N = \frac{1 - r^{N+1}}{1-r}; \text{ if } |r|<1: \lim_{N\to\infty} S_N = \frac{1 - \lim_{N\to\infty} r^{N+1}}{1-r} = \frac{1}{1-r}$$

$r = 1$: $\sum_1^{\infty} 1 = \infty$ $\qquad r = -1$: $\sum_1^{\infty} (-1)^n$ DNE

$|r| > 1$ $\quad r^n \not\to 0$ so the series doesn't converge

59.



Length of blue path:

$$\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{1 - \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2} - 1}.$$

The divergence test

$$\underbrace{\sum_{n=1}^{\infty} a_n \text{ converges}}_{} \quad \overset{\cancel{\Leftarrow}}{\Rightarrow} \quad a_n \to 0$$

Pf.

$$\Downarrow$$

$$\lim_{N\to\infty} s_N = s \qquad \underset{\downarrow s}{S_{N+1}} - \underset{\downarrow s}{S_N} = \sum_{n=1}^{N+1} a_n - \sum_{n=1}^{N} a_n = a_{N+1}$$

$$\lim_{N\to\infty} a_{N+1} = \lim_{N\to\infty} (S_{N+1} - S_N) = s - s = 0$$

$\sum_{n=1}^{\infty} \sqrt{n}$ $\qquad a_n \to \infty \Rightarrow$ series diverges

$\sum_{n=1}^{\infty} \frac{1}{n}$ $\qquad a_n \to 0$ but the series diverges

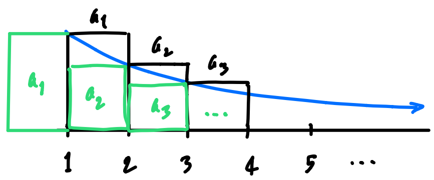
Methods for nonnegative series

The integral test

If $a_n = f(n)$, where f is nonneg., decreasing, integrable for $x \geq 1$, then

$$\int_1^\infty f(x)\,dx \quad \text{converges} \quad \text{iff} \quad \sum_{n=1}^\infty a_n \quad \text{converges}$$

iff: if and only if $\iff$



$\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ vs. $\displaystyle\int_1^\infty \frac{dx}{x^p}$

$p \neq 1$: $\displaystyle \frac{x^{-p+1}}{1-p}\Big|_1^\infty$

$p > 1$, $1 - p < 0$: finite, series conv.

$p < 1$: blows up, series div.

$p = 1$: $\ln x \Big|_1^\infty$ also blows up $\Rightarrow$ series div.

11. $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^{3/2}}$

$$\int_2^\infty \frac{dx}{x(\ln x)^{3/2}} \underset{du = \frac{dx}{x}}{\overset{u = \ln x}{=}} \int_{\ln 2}^\infty \frac{du}{u^{3/2}} = -2u^{-1/2}\Big|_{\ln 2}^\infty$$

$$= -\cancel{2/\sqrt{\infty}} + 2(\ln 2)^{-1/2} < \infty$$

$\Rightarrow$ so the series converges, by the integral test.

$$\sum \frac{1}{n(\ln n)^{3/2} + \varepsilon(n)} \qquad \varepsilon(n) = \cos\cos\cos\cos\cos n$$

The direct comparison test

If $0 \leq a_n \leq b_n$ for sufficiently large n, then

(i) $\sum b_n < \infty \Rightarrow \sum a_n < \infty$

(ii) $\sum a_n = \infty \Rightarrow \sum b_n = \infty$

59. $\displaystyle\sum_{n=2}^\infty \frac{1}{(\ln n)^4}$

$$\frac{1}{(\ln n)^4} \geq \frac{1}{n^\alpha} \qquad \frac{(\ln n)^4}{n^\alpha} \to 0 \quad (\alpha > 0)$$

so $\dfrac{(\ln n)^4}{n^\alpha} \leq 1$ for large n

choose $\alpha = 1/2$: $\dfrac{1}{(\ln n)^4} \geq \dfrac{1}{\sqrt{n}}$ eventually $\Rightarrow \sum \dfrac{1}{(\ln n)^4}$ diverges

67. $\displaystyle\sum_{n=1}^\infty \sin\frac{1}{n}$

$$\cos x \leq \frac{\sin x}{x} \qquad (|x| < \pi/2)$$

$$\frac{1}{2} \leq \cos\frac{1}{n} \leq \frac{\sin 1/n}{1/n} \Rightarrow \sin\frac{1}{n} \geq \frac{1}{2n} \quad \text{for } n \text{ large}$$

if n large

$$\Rightarrow \sum \sin\frac{1}{n} \text{ diverges}$$

$\sin x \le x \quad (x \geq 0)$

68. $\sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{\sqrt{n}}$ $\qquad \sin\frac{1}{n} \le \frac{1}{n}$ $\qquad \frac{\sin 1/n}{\sqrt{n}} \le \frac{1}{\sqrt{n}}\cdot\frac{1}{n} = \frac{1}{n^{3/2}}$

$\Rightarrow \sum \frac{\sin 1/n}{\sqrt{n}}$ converges

$\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$ converges

70. $\sum_{n=3}^{\infty} \frac{1}{e^{\sqrt{n}}}$ $\qquad \frac{1}{e^{\sqrt{n}}} \overset{!}{\le} \frac{1}{n^2}$ $\qquad \frac{n^2}{e^{\sqrt{n}}} \longrightarrow 0$ so ≤ 1 for large n

$\Rightarrow \sum \frac{1}{e^{\sqrt{n}}} < \infty$

72. $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$ $\qquad n \ln\left(1+\frac{1}{n}\right) \longrightarrow 1$ $\qquad \ln\left(1+\frac{1}{n}\right) \geq (1-\varepsilon)\frac{1}{n}$

$\left(\left(1+\frac{1}{n}\right)^n \longrightarrow e\right)$

diverges

disc. 9 (1E)

Absolutely & conditionally convergent series

Consider a series $\sum_{n=1}^{\infty} a_n$. We say the series <u>converges absolutely</u> if $\sum_{n=1}^{\infty} |a_n| < \infty$
and <u>conditionally</u> if $\sum_{n=1}^{\infty} |a_n| = \infty$ but $\sum_{n=1}^{\infty} a_n$ exists.

<u>Note</u>: for general a_n, $\sum |a_n| < \infty \Rightarrow \sum a_n$ converges.
i.e. absolute convergence implies convergence

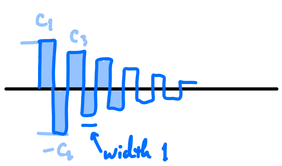
Alternating series test

The series $\sum_{n=1}^{\infty} (-1)^{n+1} c_n$ converges if <u>$c_n \geq 0$</u>, <u>c_n decreases</u>, and <u>$c_n \to 0$</u>.

Determine whether the series conv. absolutely, conditionally, or not at all:

8. $\sum_{1}^{\infty} \frac{\sin \frac{\pi n}{4}}{n^2}$ Note $\left|\frac{\sin \frac{\pi n}{4}}{n^2}\right| \leq \frac{1}{n^2}$ so $\sum \left|\frac{\sin \frac{\pi n}{4}}{n^2}\right|$ converges by the direct comparison test

hence the orig. series converges absolutely.



$c_1 - c_2 + c_3 - c_4 + c_5 - \cdots$

9. $\underbrace{\sum_{2}^{\infty} \frac{(-1)^n}{n \ln n}}$ consider $\sum_{2}^{\infty} \frac{1}{n \ln n}$: this series diverges

compare $\int_2^{\infty} \frac{dx}{x \ln x} \overset{u = \ln x}{\underset{du = \frac{dx}{x}}{=}} \int_{\ln 2}^{\infty} \frac{du}{u} = \ln \infty - \ln \ln 2$
$= \infty$

$\downarrow$

Try alternating series test

$c_n = \frac{1}{n \ln n}$

(i) $c_n \geq 0$ ✓

(ii) c_n decr. ✓ $\left(\frac{1}{x \ln x}\right)' = \frac{-(x \ln x)'}{(x \ln x)^2} = \frac{-(\ln x + 1)}{(x \ln x)^2} \leq 0$

for $x \geq 2$, $\ln x \geq \ln 2$

(iii) $c_n \to 0$ ✓

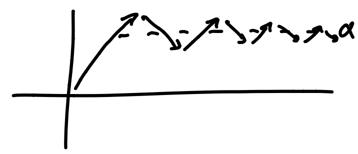
By the test, the series converges. So, #9 converges conditionally.

Rearrangement of series

Given a_n, a rearrangement is $a_{\sigma(n)}$ where σ is a permutation, i.e. $\sigma: \mathbb{N} \circlearrowleft$ is bijective.

Thm. (Riemann) Suppose $\sum a_n$ converges conditionally. Then there is a rearrangement $\sum \tilde{a}_n$ s.t. $\sum \tilde{a}_n = \alpha$ where $\alpha \in [-\infty, \infty]$ or where $\sum \tilde{a}_n$ oscillates.

Pf. sketch Consider
$$b_n = \max(a_n, 0)$$
$$c_n = -\min(a_n, 0)$$



First, observe

$$b_n - c_n = \begin{cases} a_n - 0 & a_n \geq 0 \\ 0 - (-a_n) & a_n < 0 \end{cases} = a_n$$

$$b_n + c_n = \begin{cases} a_n + 0 & a_n \geq 0 \\ 0 + (-a_n) & a_n < 0 \end{cases} = |a_n|$$

Claim $\sum b_n = \sum c_n = \infty$

If $\sum b_n = B < \infty$, $\sum c_n = \infty$:

$$\sum a_n = \sum (b_n - c_n) = B - \infty = -\infty \quad ↯ \text{ (i.e. a contradiction)}$$

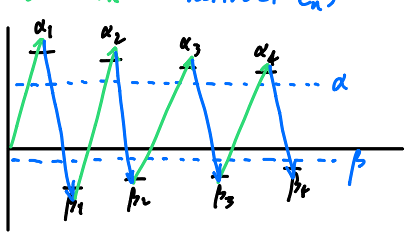
If $\sum b_n = \infty$, $\sum c_n = C < \infty$:

$$\sum a_n = \sum (b_n - c_n) = \infty - C = \infty \quad ↯$$

If $\sum b_n = B$, $\sum c_n = C$ both finite:

$$\sum |a_n| = \sum (b_n + c_n) = \sum b_n + \sum c_n = B + C < \infty \quad ↯$$

add b_n's subtract c_n's



Choose $\alpha_n \searrow \alpha$ $\qquad \alpha \geq \beta$

$\beta_n \nearrow \beta$

$\beta_{n-1} < \alpha_n$

Recall $a_n \to 0$, hence $b_n, c_n \to 0$

E.g. if $\alpha = \beta$, this shows we can rearrange in such a way that $\sum \tilde{a}_n = \alpha$.

Thm. If $\sum a_n = a$ converges absolutely, then any rearrangement converges to the same value.

The ratio test

Suppose the limit $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.

(i) If $\rho < 1$, then $\sum a_n$ converges absolutely.

(ii) If $\rho > 1$, then $\sum a_n$ diverges

(iii) If $\rho = 1$, then the test is inconclusive.

The root test

Suppose $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists.

(i) If $L < 1$, then $\sum a_n$ conv. absolutely

(ii) If $L > 1$, then $\sum a_n$ diverges

(iii) If $L = 1$, the test is inconclusive.

27. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{\cancel{(n+1)}\cancel{n!}\, n^n}{\cancel{(n+1)}(n+1)^n \cancel{n!}} = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \left(1 + \frac{1}{n}\right)^{-n} \to e^{-1}$$

as $n \to \infty$

$e^{-1} < 1$

so the series converges absolutely.

• For which x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

$$\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1} \to 0 \quad \text{for any } x$$

• How about $\sum_{n=1}^{\infty} \frac{x^n}{n}$?

$|x| < 1$ converges

$x = 1$ diverges (harmonic)

$x = -1$ converges (alternating series test)

disc. 9 (1F)

Absolutely & conditionally convergent series

We say $\sum a_n$ converges absolutely if $\sum |a_n| < \infty$

& conditionally if $\sum |a_n| = \infty$ but $\sum a_n$ converges.

Prop. Absolute convergence implies convergence.

e.g. $\sum \frac{\cos n}{n^2}$: $\sum \left|\frac{\cos n}{n^2}\right| \leq \sum \frac{1}{n^2}$.

Alternating series test

The series $\sum_{n=1}^{\infty} (-1)^{n+1} c_n$ converges if $c_n \geq 0$, c_n decreases, and $c_n \to 0$.

Determine whether the series conv. absolutely, conditionally, or not at all:

8. $\sum_1^{\infty} \frac{\sin \frac{\pi n}{4}}{n^2}$ Consider $\left|\frac{\sin \frac{\pi n}{4}}{n^2}\right| \leq \frac{1}{n^2} \Rightarrow \sum \left|\frac{\sin \pi n/4}{n^2}\right| < \infty$

↳ so this series converges absolutely.

9. $\sum_2^{\infty} \frac{(-1)^n}{n \ln n}$ Consider $\sum_2^{\infty} \frac{|(-1)^n|}{n \ln n} = \sum_2^{\infty} \frac{1}{n \ln n}$: compare to $\int_2^{\infty} \frac{dx}{x \ln x}$

$u = \ln x$

$du = \frac{dx}{x}$

$= \int_{\ln 2}^{\infty} \frac{du}{u} = \infty - \ln \ln 2 = \infty$

↳ so this series diverges

↓

✓ 1. $c_n \geq 0$

✓ 2. c_n decr. ⟶ enough to show $\left(\frac{1}{x \ln x}\right)' \leq 0$ for $x \geq 2$

✓ 3. $c_n \to 0$

$$\left(\frac{1}{x \ln x}\right)' = \frac{-(x \ln x)'}{(x \ln x)^2} = \frac{-(1 \ln x + x/x)}{(x \ln x)^2}$$

$$= \frac{-\overbrace{(\ln x + 1)}^{\geq 0 \text{ if } x \geq 2}}{\underbrace{(x \ln x)^2}_{\geq 0}}$$

so $\left(\frac{1}{x \ln x}\right)' \leq 0$ for $x \geq 2$.

So by the alternating series test, this series converges.

Rearrangement of series

Thm (Riemann) Suppose $\sum a_n$ converges conditionally. Then for $\alpha \in [-\infty, \infty]$, we can rearrange a_n into $\tilde{a}_n$ s.t. $\sum \tilde{a}_n = \alpha$, or we can get $\sum \tilde{a}_n$ to oscillate between two values.

Pf. sketch Set
$$b_n = \max(a_n, 0) \qquad c_n = -\min(a_n, 0)$$
so
$$b_n - c_n = \begin{cases} a_n - 0 & a_n \geq 0 \\ 0 - (-a_n) & a_n < 0 \end{cases} = a_n$$
$$b_n + c_n = \begin{cases} a_n + 0 & a_n \geq 0 \\ 0 + (-a_n) & a_n < 0 \end{cases} = |a_n|$$

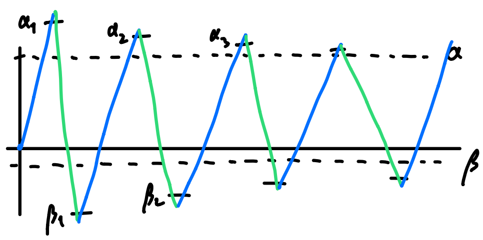
Suppose $\sum b_n = B < \infty$, $\sum c_n = \infty$:
$$\sum a_n = \sum (b_n - c_n) = B - \infty = -\infty \quad \lightning \text{ (a contradiction)}$$

Suppose $\sum b_n = \infty$, $\sum c_n = C < \infty$:
$$\sum a_n = \sum (b_n - c_n) = \infty - C = \infty \quad \lightning$$

Suppose $\sum b_n, \sum c_n < \infty$:
$$\sum |a_n| = \sum (b_n + c_n) = \sum b_n + \sum c_n < \infty \quad \lightning$$

Therefore, $\sum b_n = \sum c_n = \infty$.

Suppose $\alpha, \beta \in \mathbb{R}$.



$\alpha_j \searrow \alpha$

$\beta_j \nearrow \beta$

$\beta_{j-1} < \alpha_j$

$a_n \to 0$

Thm. Suppose $\sum a_n$ converges absolutely. Then any rearrangement $\sum a_{\sigma(n)}$ converges to the same value.

$\sum a_n = a$

$\hookrightarrow \sigma : \mathbb{N} \to \mathbb{N}$ is bijective

Pf. Choose N so large that $\sum_{n=N+1}^{\infty} |a_n| < \varepsilon$. Then
$$\left| \sum_{n=1}^{N} a_n - a \right| = \left| \sum_{n=N+1}^{\infty} a_n \right| \leq \sum_{n=N+1}^{\infty} |a_n| \leq \varepsilon.$$

Choose J so large that $\{\sigma(j) : 1 \leq j \leq J\} \supseteq \{1, \ldots, N\}$

$$\left| \sum_{j=1}^{J} a_{\sigma(j)} - a \right| \leq \left| \sum_{j=1}^{J} a_{\sigma(j)} - \sum_{j=1}^{N} a_j \right| + \underbrace{\left| \sum_{j=1}^{N} a_j - a \right|}_{\leq \varepsilon}$$

$\underbrace{a_{\sigma(1)} + \cdots + a_{\sigma(J)}}_{\text{including } a_1, \ldots, a_N} - a_1 - \cdots - a_N$

so this first term is $\leq \sum_{N+1}^{\infty} |a_j| < \varepsilon$

$\hookrightarrow$ so this is $< 2\varepsilon$

The ratio test

Suppose the limit $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.

(i) If $\rho < 1$, then $\sum a_n$ converges absolutely.

(ii) If $\rho > 1$, then $\sum a_n$ diverges

(iii) If $\rho = 1$, then the test is inconclusive.

The root test

Suppose $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists.

(i) If $L < 1$, then $\sum a_n$ conv. absolutely

(ii) If $L > 1$, then $\sum a_n$ diverges

(iii) If $L = 1$, the test is inconclusive.

27. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{\cancel{(n+1)n!}}{\cancel{n!}} \frac{n^n}{(n+1)^n} \frac{1}{\cancel{n+1}} = \left(\frac{n}{n+1}\right)^n = \left(1 + \frac{1}{n}\right)^{-n} \to e^{-1} < 1$$

so the series converges by the ratio test.

- For which x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Conv. for all x :

$$\left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0$$

- How about $\sum_{n=1}^{\infty} \frac{x^n}{n}$?

Conv. for $x \in [-1, 1)$

$$\left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = \frac{n}{n+1}|x| \to |x| < 1$$

(& check $-1, 1$ separately)

disc. 10 (1E)

- power series

$$\sum_{n=0}^{\infty} a_n x^n$$

such a series has a radius of convergence R which is equal to $\left(\lim_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1}$ if the limit exists: then the series converges in $(-R, R)$.

Ex. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad \lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$

$$R = \text{"}0^{-1}\text{"} = \infty$$

i.e. by the root test, the series converges for all x.

$$\ln\left[(n!)^{1/n}\right] = \frac{1}{n} \ln n!$$
$$= \frac{1}{n}(\ln n + \dots + \ln n/2 + \dots + \ln 1)$$
$$\geq \frac{n/2 \cdot \ln n/2}{n} = \frac{1}{2} \ln \frac{n}{2} \longrightarrow \infty$$

hence $\sqrt[n]{n!} \to \infty$.

Ex. $\sum_{n=1}^{\infty} \frac{x^n}{n}$ $\quad \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$

$\Rightarrow R = 1$ so the series converges in $(-1, 1)$

Check at $x = 1$: $\sum \frac{1}{n}$ diverges

at $x = -1$: $\sum \frac{(-1)^n}{n}$ converges $\quad$ so the series converges in $[-1, 1)$.

- Taylor series

Idea: expand $f(x) = \sum_{n=0}^{\infty} a_n x^n$ inside $(-R, R)$. What is a_n? (In general, $\sum a_n (x-c)^n$)

Thm. For $|x| < R$, we can differentiate termwise, i.e. $\frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n$

So, $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $\quad f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$, $\dots$, $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) a_n x^{n-k}$

$f(0) = a_0$, $\quad f'(0) = a_1$, $\quad f''(0) = 2 \cdot 1 \cdot a_2$, $\dots$, $\underbrace{f^{(k)}(0) = k(k-1)\dots 1 \cdot a_k}$

$$\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}$$

Ex. $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$ $\quad f^{(n)}(0) = 0$

so the series around 0 is $\sum_{0}^{\infty} \frac{0}{n!} x^n = 0$ but $f(x) > 0$ for $x > 0$ so radius of convergence is 0.

- Ex. $\frac{1}{1-x}$ at 0, 2

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad |x| < 1 \qquad x \in (-1, 1)$$

$$\frac{d^n}{dx^n} \frac{1}{1-x} = \frac{n!}{(1-x)^{n+1}}$$

$\frac{d}{dx}$: $+(1-x)^{-2}$

$\frac{d^2}{dx^2}$: $+2(1-x)^{-3}$

$\frac{d^3}{dx^3}$: $+3 \cdot 2 (1-x)^{-4}$

$\sum_{n=0}^{\infty} a_n (x-2)^n$? $\qquad a_n = \frac{f^{(n)}(2)}{n!} = \frac{n!}{(-1)^{n+1}} \cdot \frac{1}{n!} = (-1)^{n+1}$

$\hookrightarrow \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n$ $\quad$ by the root test, get convergence in $|x-2| < 1$

$\qquad \|$

$-\sum_{n=0}^{\infty} (2-x)^n$ conv. if $|2-x| < 1$ & $-\sum_{n=0}^{\infty} (2-x)^n = \frac{-1}{1-(2-x)} = \frac{1}{1-x}$.

• Differentiation of series

Ex. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\sum_{n=0}^{\infty} n x^{n-1} \Rightarrow \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

plug in $x = 1/2$: $\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{(1/2)^2} = 2$

Ex. $\sum_{n=1}^{\infty} \frac{n^2}{\pi^n} = \sum_{n=2}^{\infty} n(n-1)\left(\frac{1}{\pi}\right)^n + \sum_{n=1}^{\infty} n\left(\frac{1}{\pi}\right)^n$

Sim. to above

$$\underbrace{\frac{d^2}{dx^2}\sum_{n=0}^{\infty} x^n}_{} = \frac{2}{(1-x)^3}$$

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} \Rightarrow \sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}$$

$$= \frac{2 \cdot 1/\pi^2}{(1-1/\pi)^3} + \frac{1/\pi}{(1-1/\pi)^2}$$

<u>Probability example</u> We toss a coin repeatedly w/ prob. of heads p, $0 < p < 1$.
What is the expected number of tries until the first heads?

$$\sum_{k=1}^{\infty} kp(1-p)^{k-1} = \frac{p}{(1-(1-p))^2} = \frac{1}{p}$$

disc. 10 (1F)

• power series

Series of the form $\sum_{n=0}^{\infty} a_n x^n$.

Such a series converges in $(-R, R)$ where R is the radius of convergence. We can have $R = 0$ or ∞ as well.
If the limit exists, $R = \left(\lim_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1}$ where we use the convention $0^{-1} = \infty$ for this only.

Ex. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Claim $\underbrace{\lim_{n\to\infty} \sqrt[n]{|a_n|}}_{\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}} = 0$.

$$\ln \sqrt[n]{n!} = \frac{1}{n} \ln n! = \frac{1}{n}(\ln n + \ln(n-1) + \dots + \ln \tfrac{n}{2} + \dots + \ln 1)$$

(n even) $\geq \frac{1}{n}\left(\frac{n}{2} \ln \frac{n}{2}\right) = \frac{1}{2} \ln \frac{n}{2} \to \infty$

(n odd) $\geq \frac{1}{n}\left(\frac{n+1}{2} \ln \frac{n+1}{2}\right) \geq \frac{1}{2} \ln \frac{n}{2} \to \infty$

Ex. $\sum_{n=1}^{\infty} \frac{x^n}{n}$ $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 \Rightarrow R = 1$

$[-1, 1)$

So, series converges for all $x \in (-1, 1)$. How about endpoints?
$x = 1$: $\sum \frac{1}{n}$ diverges
$x = -1$: $\sum \frac{(-1)^n}{n}$ converges (by alternating series test)

• Taylor series (in general, $\sum a_n (x-c)^n$)

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ in $(-R, R)$. We use the following thm.

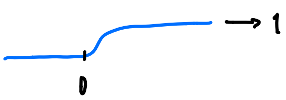
Thm. For $|x| < R$, $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

So similarly, $f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$, ..., $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k}$

At 0, $f(0) = a_0$, $f'(x) = a_1$, $f''(x) = 2 \cdot 1 \cdot a_2$, ..., $f^{(k)}(0) = \underbrace{k(k-1)\cdots 1}_{k!} \cdot a_k$

$$\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}$$

Ex. $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$



Recall from before, $f'(0) = 0$, & sim. $f^{(n)}(0) = 0$ for all $n \geq 1$.
Nevertheless, $f(x) > 0$ for $x > 0$.

The Taylor series for f about 0 is $\sum_0^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0$
so it only converges to f at $x = 0$.

• Ex. $\frac{1}{1-x}$ at 0, 2 $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$ by induction

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $|x| < 1$ $\frac{f^{(n)}(2)}{n!} = \frac{n!}{(1-2)^{n+1}} \cdot \frac{1}{n!} = (-1)^{n+1}$

$$\sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n = -\sum_{n=0}^{\infty} (2-x)^n \overset{|2-x|<1}{=} \frac{-1}{1-(2-x)} = \frac{-1}{-1+x} = \frac{1}{1-x} \quad (|x-2| < 1)$$

• Differentiation of series

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}\frac{d}{dx}x^n = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\Rightarrow \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$$

at $x = \frac{1}{2}$: $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{(1/2)^2} = 2.$

Taking a second-order derivative:

$$\frac{d^2}{dx^2}\frac{1}{1-x} = \frac{d^2}{dx^2}\sum_{n=0}^{\infty} x^n$$

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

$$\frac{2x^2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)\,x^n$$

$(|x|<1)$

Ex. $$\sum_{n=1}^{\infty}\frac{n^2}{\pi^n} = \sum_{n=1}^{\infty} n(n-1)\left(\frac{1}{\pi}\right)^n + \sum_{n=1}^{\infty} n\left(\frac{1}{\pi}\right)^n$$

$$= \frac{2(1/\pi)^2}{(1-1/\pi)^3} + \frac{1/\pi}{(1-1/\pi)^2}$$

<u>Probability example</u> We toss a coin repeatedly w/ prob. of heads p, $0<p<1$.

What is the expected $\underbrace{\text{number of tries until the first heads}}_{=:X}$?

$$P(X=n) = (1-p)^{n-1}p \qquad \underbrace{TTT\cdots T}_{n-1}H$$

By definition, $\displaystyle \mathbb{E}X = \sum_{n=1}^{\infty} n P(X=n)$

$$= \sum_{n=1}^{\infty} np(1-p)^{n-1} = p\sum_{n=1}^{\infty} n(1-p)^{n-1} = p\frac{1}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}.$$