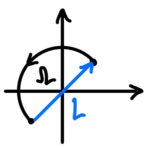
UCLA: Math 32B **Week 9 Worksheet** **Spring 20**

Recall Green's theorem: If $D \subset \mathbb{R}^2$ has boundary ∂D that is a simple closed curved oriented counterclockwise relative to D, then if $\mathbf{F} = \langle F_1, F_2 \rangle$ is a vector field with continuous partial derivative then $\oint_{\partial D} \mathbf{F} \cdot dr = \iint_D \nabla \times F \, dA$, or to write it differenly $\oint_{\partial D} F_1 dx + F_2 dy = \iint_D \partial F_2/\partial x - \partial F_1/\partial y \, dA$.

1. Let $\mathbf{F} = \langle \sin x^2, xy \rangle$ and let C be the triangle with vertices $(0,0), (3,0), (3,2)$ oriented counterclockwise. What is $\oint_C \mathbf{F} \cdot dr$?

2. Let $\mathbf{F} = \langle e^{y^2}, -e^{x^2} \rangle$ and let C be the oriented curve $r(t) = \langle \cos t, \sin t \rangle$ where $\pi/4 \le t \le 5\pi/4$. What is $\int_C \mathbf{F} \cdot dr$?



   Just doing this directly is going to leave you with a tough (but not impossible!) integral. Talk with your group about how you might go about computing this. You can't use Green's theorem since C isn't a closed curve and you can't use path independence since the vector field isn't conservative. Can you find some other curve with the same end points as C where computing the line integral is easy and then use Green's theorem? At some point you might want to use some symmetry over the line $y = -x$, i.e. you might have a region where if (x, y) is in your region then $(-y, -x)$ is as well.

3. In the examples we've done so far we've used Green's theorem to compute a line integral by realizing that the line integral is equal to an integral over a region of the plane. We can also go in the other direction, Green's theorem can help us compute double integrals. In particular it can help us integrate 1, i.e. find the area of regions of the plane.
   (a) Find the area of the ellipse $(x/a)^2 + (y/b)^2 = 1$ using Green's theorem. **Hint:** The curl of $\langle 0, x \rangle$ is 1.
   (b) Find a parameteriation of the curve $x^{2/3} + y^{2/3} = 1$ and use Green's theorem to compute the area bounded by this curve. **Hint:** Let $x(t) = \cos^3 t$. $\int_0^{2\pi} \cos^4 t \, dt = 3\pi/4$ and $\int_0^{2\pi} \cos^6 t \, dt = 5\pi/8$.

4. A few weeks ago we stated without proof that if a vector field with zero curl was defined on a simply connected region of the plane, then line integrals over it are path independent. Use Green's theorem to show that this statement is true. Make sure that you consider how to deal with the case that two curves with the same beginning and end points might intersect at other points.

1. By Green's thm.,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_T \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA$$ where T is the interior of the triangle

$$= \iint_T y - 0 \, dA = \int_0^3 \int_0^{\frac{2}{3}x} y \, dy \, dx$$

$$= \int_0^3 \frac{1}{2}\left(\frac{2}{3}\right)^2 x^2 \, dx = \frac{1}{2} \frac{2^2}{3^2} \cdot 9 = 2.$$

2. Consider Γ, obtained from C by adding the line segment L. By Green's thm.,

$$\int_C \vec{F} \cdot d\vec{r} + \underbrace{\int_L \vec{F} \cdot d\vec{r}}_{} = \iint_\Omega \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA = -\iint_\Omega \underbrace{2xe^{x^2} + 2ye^{y^2}}_{=:f} \, dx \, dy$$

$\vec{r}(t) = (t, t)$
$|t| \le \frac{1}{\sqrt{2}}$

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(e^{t^2}, -e^{t^2}\right) \cdot (1, 1)\, dt = 0$$

Since $f(-y, -x) = 2(-y)e^{(-y)^2} + 2(-x)e^{(-x)^2} = -f(x, y)$
and Ω is symm. about $y = -x$, the double integral is 0.

Thus $\displaystyle\int_C \vec{F} \cdot d\vec{r} = 0.$

3a. Let E be the ellipse, & set $p = 0$, $q = x$:

$$\int_{\partial E} p\, dx + q\, dy \underset{\text{by Green's thm.}}{=} \iint_E \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\, dA = \iint_E 1\, dA = A(E)$$

Parameterize ∂E by $(a \cos u, b \sin u)$ for $u \in [0, 2\pi]$. Then the LHS

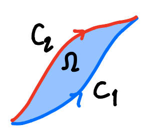
is $\displaystyle\int_0^{2\pi} (a \cos u)(b \cos u)\, du = ab \int_0^{2\pi} \sin^2 u\, du = ab\left(2\pi \cdot \frac{1}{2} - \frac{1}{2}\int_0^{2\pi} \cos 2u\, du\right).$

since $\sin^2 u = \dfrac{1 - \cos 2u}{2}$

b. Similarly w/ $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le 2\pi$, V = the region bounded by the curve,

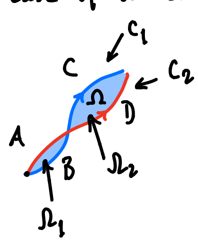
$$A(V) = \int_{\partial V} x\, dy = \int_0^{2\pi} \cos^3 t\; 3 \sin^2 t \cos t\, dt = 3\int_0^{2\pi} \cos^4 t\, (1 - \cos^2 t)\, dt$$

$$= 3\int_0^{2\pi} \cos^4 t - 3\int_0^{2\pi} \cos^6 t\, dt = 3\left(\frac{3\pi}{4}\right) - 3\left(\frac{5\pi}{8}\right) = \frac{3}{8}\pi.$$

4. 

$$\left(\int_{C_1} - \int_{C_2}\right)\vec{F} = \iint_\Omega \operatorname{curl}_z \vec{F}\, dA = 0.$$

In case of a crossing (like , not , in which case we argue as above):



$$\left(\int_{C_1} - \int_{C_2}\right)\vec{F} = \left(\int_B + \int_C - \int_A - \int_D\right)\vec{F}$$

$$= \left(\int_{B-A} - \int_{D-C}\right)\vec{F} = \iint_{\partial\Omega_1} \operatorname{curl}_z \vec{F}\, dA - \iint_{\partial\Omega_2} \operatorname{curl}_z \vec{F}\, dA$$

We obtain the result by iterating the above reasoning.