1. In this question we'll study the surface with parameterization $G(u,v) = (x(u,v), y(u,v), z(u,v))$ where:

$$x(u,v) = 2\cos u + v\sin(u/2)\cos u$$
$$y(u,v) = 2\sin u + v\sin(u/2)\sin u$$
$$z(u,v) = v\cos(u/2)$$

where $0 \le u \le 2\pi$ and $-1 \le v \le 1$.

Think of $G(u,v)$ as the sum of two vector valued functions: $c(u) = \langle 2\cos u, 2\sin u, 0\rangle$ and $s(u,v) = v\langle \sin(u/2)\cos(u), \sin(u/2)\sin u, \cos(u/2)\rangle$.

   (a) As u varies from 0 to 2π, what curve does c make?

   (b) What is the length of $s(u,v)$? for fixed u

   (c) Note that for a fixed u, $s(u,v)$ is a straight line. Note also that the projection of $s(u,v)$ into the xy-plane is parallel to $c(u,v)$. What angle does $s(u,v)$ make with the z-axis? **Hint:** Recall that the angle θ between two vectors v, w is determined by $\cos\theta = \dfrac{v \cdot w}{||v||\,||w||}$. say, for fixed $v > 0$

   (d) What surface is this? It might help to plot the points where $v = \pm 1$ and u varies from 0 to 2π in multiples of $\pi/2$.

   (e) Use this geogebra applet: `https://www.geogebra.org/m/BjV7cNwb` (or some other computer program) to visualize the surface. Be sure to reconcile what you see with your earlier work.

   (f) Compute the normal vector to this surface. What it the normal vector when $u = 0$ and what it is when $u = 2\pi$? What does this tell you?

2. What is the area of the part of the surface $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$?

3. Consider the region bounded by two cylinders of radius one intersecting at right angles to each other, i.e. $\mathcal{B} = \{(x,y,z) : x^2 + y^2 \le 1\} \cap \{(x,y,z) : x^2 + z^2 \le 1\}$. This is called a *bicylinder*.

   (a) What is the surface area of $\mathcal{B}$? **Hint:** The bycylinder's surface comes in four parts, and each has the same surface area. To find the surface area of one part you can parameterize half of one of the cylinders by thinking of it as the graph of a function, and the domain for your parameters will be constrained by the other cylinder. Once you set up the integral computing surface area the integral is quick to do.

   Mathematicians of antiquity like Archimedes knew how to compute the volume of the bicylinder– it is much easier with calculus!

   (b) What is the volume of $\mathcal{B}$? For this, it probably again is a good idea to split $\mathcal{B}$ into four or even 8 pieces.

1a. A circle of radius 2 in the xy-plane

b. $\partial_v s(u,v) = (\sin(u/2)\cos u, \sin(u/2)\sin u, \cos(u/2))$

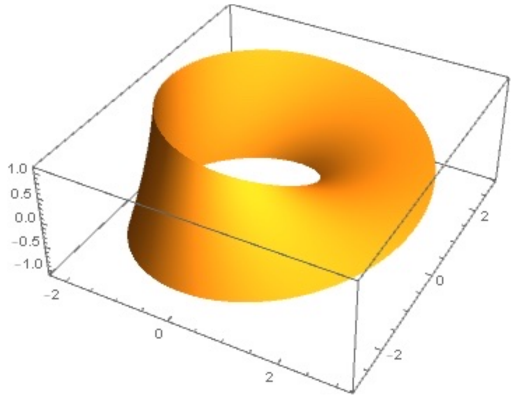
$$\|\partial_v s(u,v)\|^2 = \underbrace{\sin^2 u/2\,\cos^2 u + \sin^2 u/2\,\sin^2 u}_{\sin^2 u/2} + \cos^2 u/2 = 1$$

so the arc length is $\int_{-1}^{1} 1 = 2.$

c. For fixed u, $s(u,v)$ traces out a line segt. of length 2, centered at $c(u)$, with angle $\varphi = u/2$ down from the z-axis

d. A Möbius strip

e.



f.

$$\overrightarrow{\partial_u G} = \left(-2\sin u + \frac{v}{2}\cos\frac{u}{2}\cos u - v\sin\frac{u}{2}\sin u,\ 2\cos u + \frac{v}{2}\cos\frac{u}{2}\sin u + v\sin\frac{u}{2}\cos u,\ -\frac{v}{2}\sin\frac{u}{2}\right)$$

$$\overrightarrow{\partial_v G} = \left(\sin u/2\,\cos u,\ \sin u/2\,\sin u,\ \cos u/2\right)$$

At $u = 0,\ v = 0$: $\vec{N}(0, 0) = \overrightarrow{\partial_u G} \times \overrightarrow{\partial_v G} = (0, 2, 0) \times (0, 0, 1) = (2, 0, 0)$

$u = 2\pi,\ v = 0$: $\vec{N}(0, 2\pi) = \overrightarrow{\partial_u G} \times \overrightarrow{\partial_v G} = (0, 2, 0) \times (0, 0, -1) = (-2, 0, 0)$,

i.e. the opposite direction

2. Param. by $G(u, v) = (u, v, v^2 - u^2)$, $1 \le u^2 + v^2 \le 4$. Then

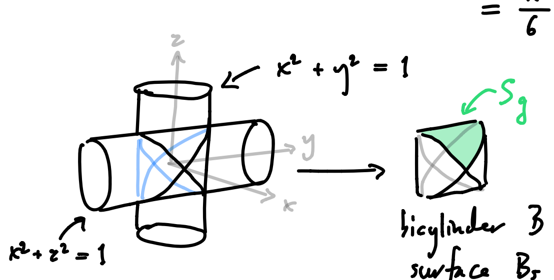
$$\frac{\partial G}{\partial u} = (1, 0, -2u),\quad \frac{\partial G}{\partial v} = (0, 1, 2v),\quad \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2u \\ 0 & 1 & 2v \end{vmatrix} = (2u, 2v, 1), \text{ so}$$

$$\iint_S dS = \iint \left\| \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right\| dA = \iint_{1 \le u^2+v^2 \le 4} \sqrt{4(u^2+v^2)+1}\, du\, dv.$$

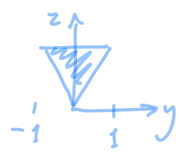
$$= 2\pi \int_1^2 \sqrt{4r^2+1}\, r\, dr = 2\pi \frac{1}{12} (4r^2+1)^{3/2} \Big|_1^2$$

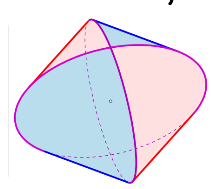
$$= \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$$

3a.



or, see this illustration from Wikipedia:





Param. the top part (green) as $\phi(y, z) = \left(\sqrt{1 - z^2}, y, z\right)$ for $|y| \le z \le 1$, $|y| \le 1$

$$\iint_{S_g} dS = \int_{-1}^{1}\int_{|y|}^{1} \| \underbrace{(0, 1, 0) \times \left(-z(1-z^2)^{-1/2}, 0, 1\right)}_{= (1,\ 0,\ z(1-z^2)^{-1/2})} \|\, dz\, dy$$

$$= \int_{-1}^{1}\int_{|y|}^{1} \sqrt{1 + z^2(1-z^2)^{-1}}\, dz\, dy$$

$$= \int_{-1}^{1}\int_{|y|}^{1} \frac{1}{\sqrt{1-z^2}}\, dz\, dy$$

$$= \int_{-1}^{1} \arcsin 1 - \arcsin |y| \, dy$$

$$= 2\int_0^1 \frac{\pi}{2} - \arcsin y \, dy = \pi - 2\int_0^1 \arcsin y \, dy = \pi - (\pi - 2) = 2$$

where we used

$$\frac{d}{dz} \arcsin z = \frac{1}{\sqrt{1-z^2}}$$

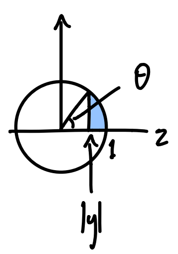
$$\& \quad \frac{d}{dy}\left(\sqrt{1-y^2} + y \arcsin y\right) = \arcsin y$$

hence $A(B_S) = 8A(S_g) = \boxed{16}$. (by symmetry)

b. Consider the region B_g given by $0 \le x \le \sqrt{1-z^2}$, $|y| \le 1$, $|y| \le z \le 1$. This gives $\frac{1}{8}$ of the (solid) bicylinder B. Thus

$$V(B) = 8V(B_g)$$

$$= 8\iiint_{B_g} dV = 8\int_{-1}^{1} \underbrace{\int_{|y|}^{1}\int_0^{\sqrt{1-z^2}} dx\, dz}_{\int_{|y|}^{1} \sqrt{1-z^2}\, dz} dy$$



where $\theta = \arccos |y|$

$$A(\text{blue}) = A(\text{sector}) - A(\text{triangle})$$

$$= \frac{1}{2}\theta - \frac{1}{2}|y| \sin\theta$$

$$= \frac{1}{2}\arccos |y| - \frac{1}{2}|y|\sqrt{1-y^2}$$

(or you can do a trig. substitution)

$$\Rightarrow V(B) = 8\int_0^1 \frac{1}{2}\arccos y - \frac{1}{2}y\sqrt{1-y^2}\, dy$$

$$= 4\underbrace{\left(y \arccos y - \sqrt{1-y^2}\right)\Big|_0^1}_{(\cancel{\arccos 1}^{\,0} - (-1))} + 4\underbrace{\int_0^1 -y\sqrt{1-y^2}\, dy}_{\frac{1}{3}(1-y^2)^{3/2}\big|_0^1} = \boxed{16/3}$$