UCLA: Math 32B **Week 7 Worksheet** **Spring 20**

1. Let $\mathbf{F}(x,y) = \langle y^2+1, 2xy-2\rangle$. Compute $\int_{\mathcal{C}} \mathbf{F}\cdot dr$ where $\mathcal{C}$ is:
   (a) The line segment from $(0,0)$ to $(1,1)$.
   (b) The path from $(0,0)$ to $(1,1)$ that first moves in a straight line to $(0,1)$ and then moves in a straight line to $(1,1)$.
   (c) Reconcile your answers with the fundamental theorem of conservative vector fields.

2. Let $f(x,y) = \sin x + x^2 y$ and let $\mathbf{F} = \nabla f$. Let $\mathcal{C}$ be the part of the parabola $y = x^2$ going from $(0,0)$ to (π, π^2).
   (a) Compute $\int_{\mathcal{C}} \mathbf{F}\cdot dr$ using the definition of vector line integrals.
   (b) Compute $\int_{\mathcal{C}} \mathbf{F}\cdot dr$ using the fundamental theorem of conservative vector fields.

3. Consider the vector field $\mathbf{F} = \langle 2xy^2z^2 + e^{x^2}, 2x^2yz^2 - e^{y^2}, 2x^2y^2z\rangle$. Let $\mathcal{C}$ be the part of the curve $\mathbf{r}(t) = \langle t, t^2, t\rangle$ $0 \le t \le 1$. Compute $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

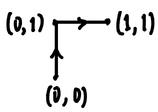
   Before you embark on doing this problem, discuss with your group different ways that you might approach this problem.

4. (To LAs: depending on how far we get on Monday I might not include this question) Find the surface area of the part of the surface $z = y^2 - x^2$ that lies between the cylinders $x^2+y^2 = 1$ and $x^2+y^2 = 4$.

5. (To LAs: if the above question is not included I'll include this one, perhaps both will be included depending on feedback from y'all and the TAs) A vector field $\mathbf{F}$ has curl zero and is defined on all of $\mathbb{R}^2$ except for $(0,0)$ and $(2,0)$.
   (a) Show that if $\mathcal{C}$ is the circle of radius R with $R > 1$ centered at $(1,0)$ then $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ is independent of R.
   (b) Integrating $\mathbf{F}$ over a small circle centered at $(0,0)$ oriented counterclockwise gives 2 and integrating $\mathbf{F}$ over a small circle centered at $(2,0)$ oriented clockwise gives -1.
   What is the integral of $\mathbf{F}$ over the circle of radius R with $R > 1$ centered at $(1,0)$ oriented counterclockwise?

1a. $\vec{r}(t) = (t, t), \quad 0 \le t \le 1$

$$\int_0^1 (t^2+1, 2t^2-2)\cdot(1,1)\,dt = \int_0^1 3t^2 - 1\,dt = t^3\Big|_0^1 - 1 = 0$$

b. 

$\vec{q}(t) = (0, t), \quad 0 \le t \le 1$

$\vec{r}(t) = (t, 1), \quad 0 \le t \le 1$

$$\int_0^1 (t^2+1, -2)\cdot(0,1)\,dt + \int_0^1 (2, 2t-2)\cdot(1,0)\,dt$$

$$= \int_0^1 -2 + \int_0^1 2 = 0$$

c. This also follows from the result since curl $\vec{F} = (\partial_x(2xy-2) - \partial_y(y^2+1))\,\hat{k} = \vec{0}$, and the region $(\mathbb{R}^2)$ is simply connected.

2a. $f = \sin x + x^2 y \qquad \vec{F} = \nabla f = (\cos x + 2xy,\ x^2)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} (\cos t + 2t^3,\ t^2) \cdot (1, 2t)\, dt$$

$$= \int_0^{\pi} \cos t + 4t^3\, dt = \pi^4$$

b. $$\int_C \vec{F} \cdot d\vec{r} = f(\pi, \pi^2) - f(0,0) = \sin \pi + \pi^2 \pi^2 - 0 = \pi^4$$

3. $$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy^2z^2 + e^{x^2} & 2x^2yz^2 - e^{y^2} & 2x^2y^2z \end{vmatrix}$$

$$= (4x^2yz - 4x^2yz,\ 4xy^2z - 4xzy^2,\ 4xyz^2 - 4xyz^2) = \vec{0}$$

so since $\vec{F}$ is defined on $\mathbb{R}^3$, which is simply connected, $\vec{F}$ is conservative.

We can thus consider $\vec{r}(t) = (t, t, t)$, $0 \le t \le 1$, instead.

So $$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (2t^5 + e^{t^2},\ 2t^5 - e^{t^2},\ 2t^5) \cdot (1, 1, 1)\, dt$$

$$= \int_0^1 2t^5 + \cancel{e^{t^2}} + 2t^5 - \cancel{e^{t^2}} + 2t^5\, dt = \int_0^1 6t^5\, dt = 1$$

4. Param. by $G(u,v) = (u, v, v^2 - u^2)$, $1 \le u^2 + v^2 \le 4$. Then

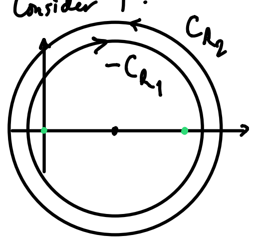
$$\frac{\partial G}{\partial u} = (1, 0, -2u), \quad \frac{\partial G}{\partial v} = (0, 1, 2v), \quad \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2u \\ 0 & 1 & 2v \end{vmatrix} = (2u, 2v, 1), \text{ so}$$

$$\iint_S dS = \iint \left\| \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right\| dA = \iint_{1 \le u^2+v^2 \le 4} \sqrt{4(u^2+v^2)+1}\, du\, dv.$$

$$= 2\pi \int_1^2 \sqrt{4r^2+1}\, r\, dr = 2\pi \frac{1}{12} (4r^2+1)^{3/2} \Big|_1^2$$

$$= \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$$

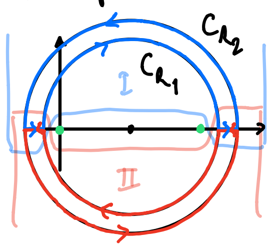
5a. Consider Γ:



$(R_2 > R_1 > 1)$.

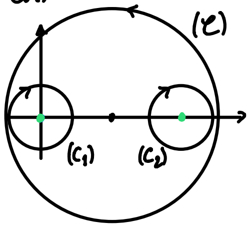
Then $$\int_{C_{R_2}} \vec{F} \cdot d\vec{r} = \int_{\Gamma} \vec{F} \cdot d\vec{r} + \int_{C_{R_1}} \vec{F} \cdot d\vec{r}.$$

splitting Γ above and below the x-axis:

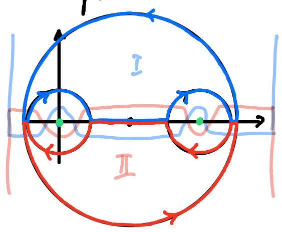


Since $\vec{F}$ is conservative in the regions I, II, the integrals over the blue & red curves are 0. Hence $\int_\Gamma \vec{F}\cdot d\vec{r} = 0$, so $\int_{C_{R_2}} \vec{F}\cdot d\vec{r} = \int_{C_{R_1}} \vec{F}\cdot d\vec{r}$.

b. Consider Γ:



Decompose into two curves



Since $\vec{F}$ is conservative in the regions I, II, the integrals are 0.

Hence

$$0 = \int_\Gamma \vec{F}\cdot d\vec{r} = \int_{C_1} + \int_{C_2} + \int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = -2 + +(-1) + \int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$$

from orientation of the circles

$$\Rightarrow \int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = 3.$$