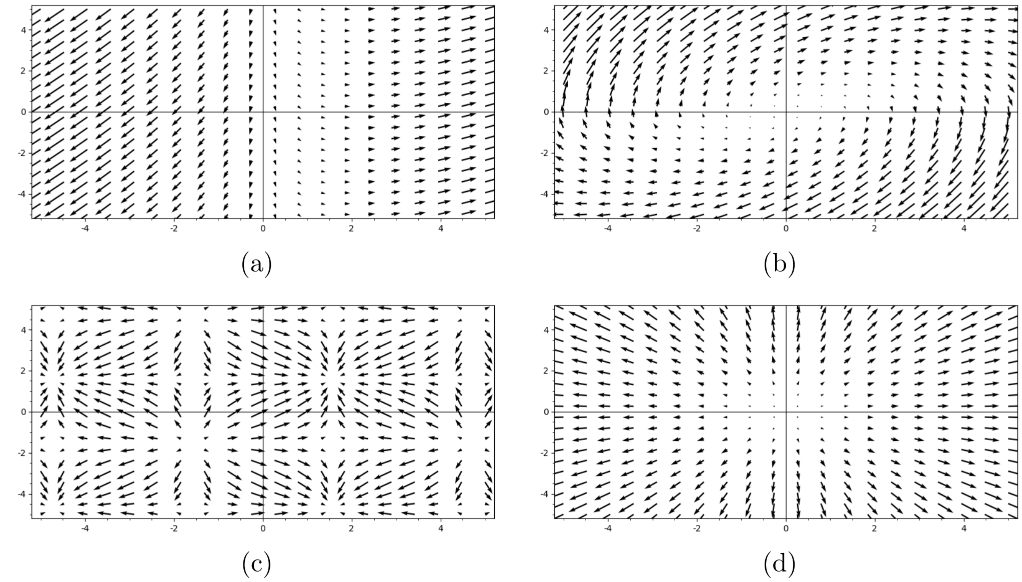

Figure 1

1. Match the vector field with its plot in the above figure. Talk with your group and relate characteristics of the plots to the functions.
   (a) $\mathbf{F}(x,y) = \langle x, y\rangle$ d
   (b) $\mathbf{F}(x,y) = \langle y, y-x\rangle$ b
   (c) $\mathbf{F}(x,y) = \langle \cos x, \cos y\rangle$ c
   (d) $\mathbf{F}(x,y) = \langle x, x-2\rangle$ a

2.
(a) $\phi(x,y) = \frac{1}{2}(x^2+y^2)$ is a potential fn.
(c) $\phi(x,y) = \sin x + \sin y$ is a potential fn.
(b) curl $\vec{F} = -2\hat{k} \neq \vec{0}$ so not conservative
(d) curl $\vec{F} = \hat{k} \neq \vec{0}$ so $\vec{F}$ is not conservative

2. Which of the above vector fields can you conclude are *not* conservative? For the others, can you find a potential function?

3. Show that if $\mathbf{F}(x,y,z)$ is a vector field with smooth component functions then the divergence of the curl of $\mathbf{F}$ is 0, i.e. $\nabla \cdot (\nabla \times F) = 0$. Give an explanation for why this is not surprising in light of what you know about $\cdot$ and $\times$ from 32a.

4. Try to determine whether or not the vector field $\mathbf{F}(x,y,z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z}\rangle$ is conservative. If you think it is conservative, find a potential function.

5. Consider the vector field $\mathbf{F}(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$.
   (a) Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is the circle of radius r centered at the origin, oriented counterclockwise.
   (b) Find parameterized curves $r(t)$ so that $\mathbf{F}(r(t)) = r'(t)$ (such curves are called *flow lines* of the vector field). If this vector field was describing motion in a fluid, flow lines describe how a particle would move if it was released into the fluid.
   $\boxed{!}$ (c) Show that the curl of $\mathbf{F}$ is 0. Is this surprising in light of numbers 2 and 4 on this worksheet (and what we've talked about in class, if you are doing this worksheet later in the week)? Give a resolution to the conundrum. **Hint:** What are the domains of all these vector fields?

3. $$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\partial_y F_3 - \partial_z F_2,\ -(\partial_x F_3 - \partial_z F_1),\ \partial_x F_2 - \partial_y F_1\right),$$ so

$$\nabla \cdot (\nabla \times \vec{F}) = \cancel{\partial_x \partial_y F_3} - \cancel{\partial_x \partial_z F_2} - \cancel{\partial_y \partial_x F_3} + \cancel{\partial_y \partial_z F_1} + \cancel{\partial_z \partial_x F_2} - \cancel{\partial_z \partial_y F_1} = 0$$

4. $\int y^2\,dx = xy^2 + f(y,z)$ $\qquad F_1 = \frac{\partial \phi}{\partial x}, \quad F_2 = \frac{\partial \phi}{\partial y}, \quad F_3 = \frac{\partial \phi}{\partial z}$

$\int 2xy + e^{3z}\,dy = xy^2 + ye^{3z} + g(x,z)$

$\phi(x,y,z) = xy^2 + ye^{3z}$

$\int 3ye^{3z}\,dz = ye^{3z} + h(x,y)$

5a. $\vec{r}(t) = (r\cos t,\ r\sin t) \qquad (0 \le t \le 2\pi)$

$$\int \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi}\left(-\frac{r\sin t}{r^2},\ \frac{r\cos t}{r^2}\right)\cdot \underbrace{(-r\sin t,\ r\cos t)}_{\vec{r}\,'(t)}\,dt$$

$$= \int_0^{2\pi} \frac{r^2\overbrace{(\cos^2 t + \sin^2 t)}^{=1}}{r^2}\,dt = 2\pi$$

b. $F(x,y) = (x', y')$ For this to make sense, need $x^2+y^2>0$.

Then $\quad x' = \dfrac{-y}{x^2+y^2}, \quad y' = \dfrac{x}{x^2+y^2} \quad \Rightarrow x^2+y^2 = \dfrac{x}{y'}$

$\Rightarrow \dfrac{x'}{y'} = -\dfrac{y}{x}$ so e.g. $x(t) = \cos t,\ y(t) = \sin t$ works.

More generally for $r>0$, we can do $(x,y) = (r\cos\frac{t}{r^2},\ r\sin\frac{t}{r^2})$.

c. $$\operatorname{curl}\vec{F} = \underbrace{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix}}_{\nabla = (\partial_x, \partial_y, \partial_z)} = \left(\partial_x\left(\frac{x}{x^2+y^2}\right) - \partial_y\left(\frac{-y}{x^2+y^2}\right)\right)\hat{k}$$

$$= \left(\frac{(x^2+y^2) - x(2x)}{x^2+y^2} + \frac{(x^2+y^2) - y(2y)}{x^2+y^2}\right)\hat{k}$$

$$= \vec{0}$$

This is not a contradiction, b/c the converse of

"$\vec{F}$ conservative $\Rightarrow$ curl $\vec{F} = \vec{0}$"

is false!

$\cancel{\Leftarrow}$