B. Stone

WORKSHEET
DISCUSSION SECTION 9
DUE 6/3 AT MIDNIGHT PDT

(1) Suppose $u = u(x, y)$, $x = r\cos(\theta)$, and $y = r\sin(\theta)$.
  (a) Use the chain rule to calculate $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$.
  (b) Use your formulas in part (a) to show that
$$\|\nabla u\|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$
Now let $u(x, y) = xy$.
  (c) What is $u(x(r, \theta), y(r, \theta))$?
  (d) Use your equation in part (b) to compute $\|\nabla u\|^2$.
  (e) Now compute $\|\nabla u\|^2$ using the variables x and y. Make sure your answer agrees with what you got in part (d).

(2) Suppose $F(x, y, z)$ is a function, and consider the equation $F(x, y, z) = 0$. In many cases, this equation implicitly defines z as a function of x and y, y as a function of x and z, and x as a function of y and z. People sometimes write
$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{\partial}{\partial y}\Big[x(y, z)\Big]$$
for the y-partial derivative of $x = x(y, z)$, with z being held constant.
  (a) Compute $(\partial x/\partial y)_z$ using implicit differentiation. Hint: differentiate the equation $F(x(y, z), y, z) = 0$ and solve for $(\partial x/\partial y)_z$.
  (b) Do the same for $(\partial z/\partial x)_y$ and $(\partial y/\partial z)_x$.
  (c) Show that the *cyclic relation* below holds.
$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$
  (d) The *ideal gas law* states that $PV = nRT$, where P is pressure, V is volume, T is temperature, and n and R are constants. Reformulate the ideal gas law in the form $F = 0$ and verify the cyclic relation.

1a. $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$\frac{\partial u}{\partial y}(r\cos\theta, r\sin\theta)$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} r\cos\theta$$

b. $$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)^2 + \cancel{\frac{1}{r^2}}\left(-\frac{\partial u}{\partial x}\cancel{r}\sin\theta + \frac{\partial u}{\partial y}\cancel{r}\cos\theta\right)^2$$
$$= \left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta + \cancel{2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta} + \left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta$$
$$+ \left(\frac{\partial u}{\partial x}\right)^2\sin^2\theta - \cancel{2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta} + \left(\frac{\partial u}{\partial y}\right)^2\cos^2\theta$$
$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

c. $u(x(r,\theta), y(r,\theta)) = x(r,\theta)\,y(r,\theta) = r^2\cos\theta\sin\theta$

d. $$\|\nabla u\|^2 = (2r\cos\theta\sin\theta)^2 + \frac{1}{r^2}\left(r^2(\cos^2\theta - \sin^2\theta)\right)^2$$
$$= 4r^2\cos^2\theta\sin^2\theta + r^2(\cos^2\theta - \sin^2\theta)^2$$
$$= r^2(4\cos^2\theta\sin^2\theta + \cos^4\theta + \sin^4\theta - 2\sin^2\theta\cos^2\theta)$$
$$= r^2(\cos^2\theta + \sin^2\theta)^2 = r^2$$

e. $$\|\nabla u\|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = y^2 + x^2 = r^2$$

2a. $$\underbrace{\frac{\partial}{\partial y}F(x(y,z),y,z)}_{\text{by the chain rule}} = \frac{\partial}{\partial y}0 = 0$$

$$\frac{\partial F}{\partial x}\left(\frac{\partial x}{\partial y}\right)_z + \frac{\partial F}{\partial y}\cancelto{1}{\frac{\partial y}{\partial y}} + \frac{\partial F}{\partial z}\cancelto{0}{\frac{\partial z}{\partial y}} \Rightarrow \frac{\partial F}{\partial x}\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\partial F}{\partial y}$$

since z is const.

$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = \frac{-\partial F/\partial y}{\partial F/\partial x}$$

b. Similarly $$\left(\frac{\partial z}{\partial x}\right)_y = \frac{-\partial F/\partial x}{\partial F/\partial z}$$
$$\left(\frac{\partial y}{\partial z}\right)_x = \frac{-\partial F/\partial z}{\partial F/\partial y}$$

so

c. $$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = \left(-\frac{\cancel{\partial F/\partial y}}{\cancel{\partial F/\partial x}}\right)\left(-\frac{\cancel{\partial F/\partial z}}{\cancel{\partial F/\partial y}}\right)\left(-\frac{\cancel{\partial F/\partial x}}{\cancel{\partial F/\partial z}}\right) = -1.$$

d. $F = PV - nRT = 0$

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{\partial F/\partial V}{\partial F/\partial P} = -1, \quad \left(\frac{\partial V}{\partial T}\right)_P = -\frac{\partial F/\partial T}{\partial F/\partial V} = 1, \quad \left(\frac{\partial T}{\partial P}\right)_V = -\frac{\partial F/\partial P}{\partial F/\partial T} = 1$$

$$\Rightarrow \left(\frac{\partial P}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_P\left(\frac{\partial T}{\partial P}\right)_V = -1$$