B. Stone

WORKSHEET
DISCUSSION SECTION 8
DUE 5/27 AT MIDNIGHT PDT

(1) The *error* of an estimate is

$$\text{actual value} - \text{estimated value.}$$

The *percent error* of an estimate is

$$\left|\frac{\text{error}}{\text{actual value}}\right| \cdot 100$$

(a) Compute the linearization $L(x,y)$ of $f(x,y) = xy$ at (a,b).
(b) Estimate $f(1.1 \cdot a, 1.1 \cdot b)$ using your linearization at (a,b).
(c) Let $\varepsilon(a,b)$ be the percent error between $L(1.1 \cdot a, 1.1 \cdot b)$ and $f(1.1 \cdot a, 1.1 \cdot b)$. What is the value of $\varepsilon(a,b)$?
(d) What is

$$\lim_{(a,b)\to(\infty,\infty)} \varepsilon(a,b)?$$

(e) Compare your previous answers to the percent error between $L(a+10, b+10)$ and $f(a+10, b+10)$ as $(a,b) \to (\infty,\infty)$.
(f) Why are your answers in parts (d) and (e) different?

(2) This exercise unpacks the sense in which the directional derivative is the slope of a tangent line. Suppose $f(x,y)$ is a two-variable function, (a,b) is a point in the 2D plane, and $\langle h,k\rangle$ is a unit vector.

(a) Parametrize the line in the xy plane that passes through the point (a,b) and is parallel to the vector $\langle h,k\rangle$.
(b) Find an equation for the plane P in 3D space that passes through $(a,b,f(a,b))$, and which is parallel to the vectors $\langle h,k,0\rangle$ and $\langle 0,0,1\rangle$.
(c) Let $\mathcal{C}$ be the intersection of the plane P with the graph of $f(x,y)$. Find a parametrization of $\mathcal{C}$ of the form

$$\vec{r}(t) = \langle a+th, b+tk, z(t)\rangle.$$

(d) Use the limit definition of the derivative to compute $\frac{d}{dt}\vec{r}(0)$. Your answer should involve the directional derivative $D_{\langle h,k\rangle}f(a,b)$.
(e) Interpret $\frac{d}{dt}\vec{r}(0)$ as the slope of a line through $(a,b,f(a,b))$. What does this tell you about $D_{\langle h,k\rangle}f(a,b)$?

1 a. $f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = ab + b(x-a) + a(y-b)$

b. $f(1.1a, 1.1b) \approx L(1.1a, 1.1b) = ab + b(0.1a) + a(0.1b) = 1.2ab$

c. $\varepsilon(a,b) = \left|\frac{L(1.1a, 1.1b) - f(1.1a, 1.1b)}{f(1.1a,1.1b)}\right| \cdot 100 = \frac{|1.2 - 1.1^2|ab}{1.1^2 ab} \cdot 100 \approx 0.826 \quad (a, b \neq 0)$

d. The function ε is const. so the limit is the value in (c)

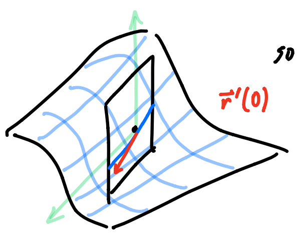
e. $L(a+10, b+10) = ab + b \cdot 10 + a \cdot 10$

$$\left| \frac{L(a+10, b+10) - f(a+10, b+10)}{f(a+10, b+10)} \right| \cdot 100 = \frac{100^2}{(a+10)(b+10)} \to 0$$

f. In (e), asymptotically $(a+10, b+10) \sim (a, b)$ as $a, b \to \infty$
This is not the case in (d) because of the scaling factor.

2a. $\vec{L}(t) = (a, b, 0) + t(h, k, 0) \quad (t \in \mathbb{R})$

b. $\vec{n} = (h, k, 0) \times (0, 0, 1) = (k, -h, 0)$



so the plane is
$$kx - hy = ka - hb$$

c. $\vec{r}(t) = (a + th, b + tk, f(a+th, b+tk))$

d.
$$\vec{r}\,'(0) = \lim_{\varepsilon \to 0} \frac{(a+\varepsilon h, b+\varepsilon k, f(a+\varepsilon h, b+\varepsilon k)) - (a, b, f(a, b))}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \left(h, k, \frac{f(a+\varepsilon h, b+\varepsilon k) - f(a, b)}{\varepsilon}\right)$$

$$= (h, k, D_{(h,k)} f(a, b))$$

e. It gives the slope of the tangent line in the direction (h, k).