WORKSHEET
DISCUSSION SECTION 6
DUE 5/13 AT MIDNIGHT PDT

(1) Consider the function

$$f(x,y) = \frac{x^2 y}{x^4 + y^2}.$$

This is a standard example that shows how subtle 2D limits can be.

(a) Let $y = mx$ for some constant m. Compute the limit

$$\lim_{x \to 0} \frac{x^2 y}{x^4 + y^2}.$$

This shows that the limit of $f(x,y)$ exists as we approach 0 along any line.

(b) Let $y = mx^2$ for some constant m. Compute the limit

$$\lim_{x \to 0} \frac{x^2 y}{x^4 + y^2}.$$

(c) Are your answers in (b) and (a) the same? Does the 2D limit

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2}$$

exist? Why or why not?

(d) (**Optional**) Find a function $f(x,y)$ such that the limit as (x,y) approaches $(0,0)$ along any line mx or parabola mx^2 is 0, but which has no 2D limit at $(0,0)$.

(2) The *Frenet-Serret formulas* are a set of differential equations satisfied by the unit tangent $\vec{T}$, the unit normal $\vec{N}$, and the binormal $\vec{B}$.[1] We have seen two in the definitions of curvature and torsion, namely:

$$\frac{d\vec{T}}{ds} = \kappa \vec{N}, \quad \text{so that } \kappa = \left\| \frac{d\vec{T}}{ds} \right\|, \quad \text{and} \quad \frac{d\vec{B}}{ds} = -\tau \vec{N}.$$

(Note: this is for an arc-length parameterization)

This exercise will outline a calculation of $d\vec{N}/ds$. You may assume the following:

There are unique scalars a, b, and c (for each s) such that $d\vec{N}/ds = a\vec{T} + b\vec{N} + c\vec{B}$.

(a) Explain why $\vec{T} \bullet \vec{N} = 0$.

(b) Use the product rule to differentiate $\vec{T} \bullet \vec{N}$, and show that $\vec{T} \bullet d\vec{N}/ds = -\kappa$.

(c) Use the distributivity of $\bullet$ over addition to show that $\vec{T} \bullet d\vec{N}/ds = a$.

(d) Conclude that $a = -\kappa$.

(e) Differentiate $\vec{N} \bullet \vec{N}$ and imitate (a)–(d) to show that $b = 0$.

(f) Differentiate $\vec{B} \bullet \vec{N}$ to show that $c = \tau$.

In summary, $d\vec{N}/ds = -\kappa \vec{T} + \tau \vec{B}$.

$$\vec{N}'(s) = a(s)\vec{T}(s) + b(s)\vec{N}(s) + c(s)\vec{B}(s)$$

$$\underbrace{\vec{T} \cdot \vec{N}'}_{a} = a \underbrace{\vec{T} \cdot \vec{T}}_{= \|\vec{T}\|^2 = 1} + b \underbrace{\cancel{\vec{T} \cdot \vec{N}}}_{0} + c \underbrace{\cancel{\vec{T} \cdot \vec{B}}}_{0}$$

[1] Recall that $\vec{T}$, $\vec{N}$, and $\vec{B}$ are unit vectors that are pairwise orthogonal, just like the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$.

1a. w/ $y = mx$,

$$\frac{x^2 y}{x^4 + y^2} = \frac{x^2 mx}{x^4 + m^2 x^2} \overset{x \neq 0}{=} \frac{mx}{x^4 + m^2} \longrightarrow 0 \quad \text{as } x \to 0.$$

b. w/ $y = mx^2$,

$$\frac{x^2 y}{x^4 + y^2} = \frac{x^2 mx^2}{x^4 + m^2 x^4} \overset{x \neq 0}{=} \frac{m}{1 + m^2} \quad \text{dep. on } m$$

c. No, so that the limit doesn't exist.

d. <u>Ex. 1.</u> $f(x, y) = \dfrac{x^3 y}{x^6 + y^2}$

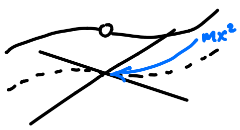
so

$$\frac{x^3 mx}{x^6 + (mx)^2} \overset{x \neq 0}{=} \frac{mx^2}{x^4 + m^2} \longrightarrow 0$$

$$\frac{x^3 mx^2}{x^6 + (mx^2)^2} \overset{x \neq 0}{=} \frac{mx}{x^2 + m^2} \longrightarrow 0$$

$$\frac{x^3 mx^3}{x^6 + (mx^3)^2} \overset{x \neq 0}{=} \frac{m}{1 + m^2} \not\longrightarrow 0 \quad \text{for } m \neq 0$$

so the 2D limit doesn't exist



<u>Ex. 2.</u> $f(x, y) = \begin{cases} 1 & \text{if } y = x^3,\ x \neq 0 \\ 0 & \text{oth.} \end{cases}$

One can check that this ad-hoc function also has the desired properties.

2a. $(\vec{T} \cdot \vec{N})(s) = \vec{T}(s) \cdot \dfrac{\vec{T}'(s)}{\|\vec{T}'(s)\|} = 0$

since $\|\vec{T}(s)\|^2 = 1 \implies \dfrac{d}{ds}\|\vec{T}(s)\|^2 = 0 \implies 2\vec{T}(s) \cdot \vec{T}'(s) = 0.$

b. $$\frac{d}{ds}\underbrace{(\vec{T} \cdot \vec{N}(s))}_{=0} = \underbrace{\vec{T}'(s) \cdot \vec{N}(s)}_{=\kappa \text{ since } \vec{T}'(s) = \kappa \vec{N}(s)} + \vec{T}(s) \cdot \vec{N}'(s) \implies \vec{T}(s) \cdot \vec{N}'(s) = -\kappa$$

c. We have $\vec{T} \cdot \vec{N}'(s) = \vec{T} \cdot (a\vec{T} + b\vec{N} + c\vec{B}) = a\vec{T} \cdot \vec{T} + b\cancelto{0}{\vec{T} \cdot \vec{N}} + c\cancelto{0}{\vec{T} \cdot \vec{B}} = a$

d. hence $a = -\kappa$ by part (b).

e. Similarly, $0 = (\vec{N} \cdot \vec{N})'(s) = 2\vec{N}(s) \cdot \vec{N}'(s)$ so $b = \vec{N} \cdot \vec{N}'(s) = 0$.

f. $\frac{d}{ds}\underbrace{(\vec{B} \cdot \vec{N})}_{=0}(s) = \underbrace{\vec{B}'(s)}_{=-\tau\vec{N}(s)} \cdot \vec{N}(s) + \vec{B}(s) \cdot \vec{N}'(s) \Rightarrow c = \vec{B}(s) \cdot \vec{N}'(s) = \tau$

We conclude that

$$\frac{d\vec{N}}{ds} = -\kappa\vec{T} + \tau\vec{B}.$$

<u>Prop.</u> $\|\vec{r}\| = \mu \Rightarrow \vec{r} \cdot \vec{r}' = 0$ (we can use this for 2(a).)

$$\|\vec{r}(t)\|^2 = \vec{r}(t) \cdot \vec{r}(t)$$

$$0 = \frac{d}{dt}\|\vec{r}(t)\|^2 = \vec{r} \cdot \vec{r}' + \vec{r}' \cdot r = 2\vec{r} \cdot \vec{r}'$$

$$\Rightarrow \vec{r}(t) \cdot \vec{r}'(t) = 0$$