WORKSHEET
DISCUSSION SECTION 5
DUE 5/6 AT MIDNIGHT PDT

(1) Consider the logarithmic spiral parametrized by $\vec{r}(t) = e^t\langle\cos(t), \sin(t)\rangle$.
  (a) Show that $\left\|\frac{d}{dt}\vec{r}(t)\right\| = e^t\sqrt{2}$.
  (b) Show that
$$\lim_{t\to-\infty}\vec{r}(t) = \langle 0, 0\rangle.$$
  Is there any real number t such that $\vec{r}(t) = \langle 0, 0\rangle$? Why or why not?
We like to think of $\vec{r}(t)$ as emerging from the origin, so we'll use $-\infty$ as our lower bound in arc length integrals.
  (c) Compute the arc length integral
$$s = \int_{-\infty}^{t}\left\|\frac{d}{du}\vec{r}(u)\right\| du.$$
  This expresses s as a function of t.
  (d) Keep s and t as in (c). Express t as a function of s.
  (e) Compute the arc length parametrization $\vec{r}_1(s) = \vec{r}(t(s))$ of the logarithmic spiral.
  (f) Check that $\frac{d}{ds}\|\vec{r}_1(s)\| = 1$, i.e. that $\vec{r}_1(s)$ really is an arc length parametrization.

(2) Consider the Archimedean spiral parametrized by $\vec{r}(t) = t\langle\cos t, \sin t\rangle$.
  (a) Compute the first and second derivatives of $\vec{r}(t)$.
  (b) Show that the curvature of $\vec{r}(t)$ is
$$\kappa(t) = \frac{t^2+2}{t^2+1}\cdot\frac{1}{\sqrt{t^2+1}}.$$
  It may be helpful to recall (from homework) that the speed of $\vec{r}(t)$ is $\sqrt{t^2+1}$.
  (c) Let $c(t) = \frac{1}{t}$ be the curvature of a circle with radius t. Show that
$$\lim_{t\to\infty}\frac{\kappa(t)}{c(t)} = 1.$$
  (d) (**Optional**) What does (c) tell you circles and Archimedean spirals?

1a. $\vec{r}'(t) = (e^t\cos t - e^t\sin t,\ e^t\sin t + e^t\cos t)$

$$\|\vec{r}'(t)\|^2 = e^{2t}\left((\cos t - \sin t)^2 + (\sin t + \cos t)^2\right)$$
$$= e^{2t}\cdot 2\underbrace{(\cos^2 t + \sin^2 t)}_{=1} \implies \|\vec{r}'(t)\| = \sqrt{2}\,e^t$$

b. Since $|\cos t|, |\sin t| \leq 1$,
we have $0 \leq |e^t\cos t| \leq e^t$, $0 \leq |e^t\sin t| \leq e^t$, hence $e^t\cos t, e^t\sin t \to 0$ as $t \to -\infty$ by the squeeze[1] lemma.
Since $\|\vec{r}(t)\| = e^t > 0$, there is no $t \in \mathbb{R}$ for which $\vec{r}(t) = \vec{0}$.

c. $s(t) = \int_{-\infty}^{t} \|\vec{r}\,'(u)\|\, du = \int_{-\infty}^{t} \sqrt{2} e^{u}\, du = \sqrt{2} e^{u} \Big|_{-\infty}^{t} = \sqrt{2} e^{t}$

d. $t(s) = \log \frac{s}{\sqrt{2}} \quad (s > 0)$

e. $\vec{r}_1(s) = \vec{r}(t(s)) = e^{\log s/\sqrt{2}} (\cos \log s/\sqrt{2}, \sin \log s/\sqrt{2})$

$$= \frac{s}{\sqrt{2}} \left(\cos \log \frac{s}{\sqrt{2}}, \sin \log \frac{s}{\sqrt{2}} \right)$$

f. $\vec{r}_1\,'(s) = \Big(\frac{1}{\sqrt{2}} \cos \log \frac{s}{\sqrt{2}} - \cancel{\frac{s}{\sqrt{2}}} \sin \log \frac{s}{\sqrt{2}} \cdot \cancel{\frac{1}{s/\sqrt{2}}} \cdot \frac{1}{\sqrt{2}},$

$\frac{1}{\sqrt{2}} \sin \log \frac{s}{\sqrt{2}} + \cancel{\frac{s}{\sqrt{2}}} \cos \log \frac{s}{\sqrt{2}} \cdot \cancel{\frac{1}{s/\sqrt{2}}} \cdot \frac{1}{\sqrt{2}} \Big)$

$$\|\vec{r}_1\,'(s)\|^2 = \frac{1}{2} \left(\cos \log \frac{s}{\sqrt{2}} - \sin \log \frac{s}{\sqrt{2}} \right)^2 + \frac{1}{2} \left(\sin \log \frac{s}{\sqrt{2}} + \cos \log \frac{s}{\sqrt{2}} \right)^2$$

$$= \frac{1}{2} \Big(\underbrace{\cos^2 \log \frac{s}{\sqrt{2}} + \sin^2 \log \frac{s}{\sqrt{2}}}_{1} - \cancel{2 \sin \log \frac{s}{\sqrt{2}} \cos \log \frac{s}{\sqrt{2}}}$$

$$+ \underbrace{\sin^2 \log \frac{s}{\sqrt{2}} + \cos^2 \log \frac{s}{\sqrt{2}}}_{1} + \cancel{2 \sin \log \frac{s}{\sqrt{2}} \cos \log \frac{s}{\sqrt{2}}} \Big)$$

$$= 1.$$

2a. $\vec{r}\,'(t) = (\cos t - t \sin t, \ \sin t + t \cos t)$

$$\vec{r}\,''(t) = (-\sin t - (\sin t + t \cos t), \ \cos t + (\cos t - t \sin t))$$

$$= (-2 \sin t - t \cos t, \ 2 \cos t - t \sin t)$$

b. We will need:

$$\|\vec{r}\,'(t)\| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2} = \sqrt{1 + t^2}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos t - t \sin t & \sin t + t \cos t & 0 \\ -2 \sin t - t \cos t & 2 \cos t - t \sin t & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \cos t - t \sin t & \sin t + t \cos t \\ -2 \sin t - t \cos t & 2 \cos t - t \sin t \end{vmatrix} \hat{k}$$

$$= \big((\cos t - t \sin t)(2 \cos t - t \sin t) - (\sin t + t \cos t)(-2 \sin t - t \cos t) \big) \hat{k}$$

$$= (2 + t^2) \hat{k}$$

$$\Rightarrow \|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = 2 + t^2$$

$$\Rightarrow \quad \kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{2 + t^2}{(1 + t^2)^{3/2}}, \quad \text{as desired.}$$

c. $$\lim_{t \to \infty} \frac{\kappa(t)}{C(t)} = \lim_{t \to \infty} t \cdot \frac{2 + t^2}{(1 + t^2)^{3/2}} = \lim_{t \to \infty} \frac{t}{\sqrt{1 + t^2}} \cdot \frac{2 + t^2}{1 + t^2}$$

$$= \lim_{t \to \infty} \frac{1}{\sqrt{1/t^2 + 1}} \cdot \frac{2/t^2 + 1}{1/t^2 + 1} = 1.$$

d. Asymptotically, the curvature of the spiral matches that of the circle.
We can write:

$$\kappa \sim C \quad \text{as } t \to \infty.$$