Solutions

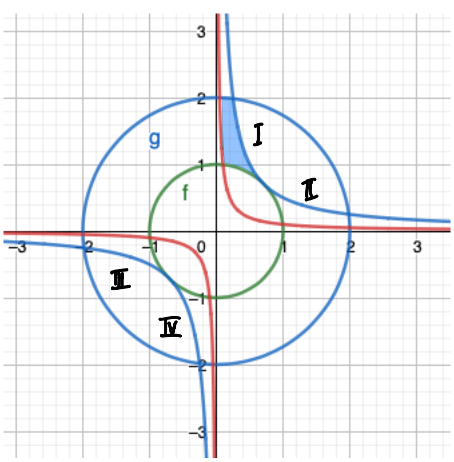
1. (a) Find a change of coordinates map G that takes the unit square $[0,1]\times[0,1]$ to the parallelogram with vertices $(0,0),(2,1),(1,2)(3,3)$.
   (b) Find the Jacobian of G.
   (c) Find a change of coordinates map G' that takes the unit square $[0,1]\times[0,1]$ to the parallelogram with vertices $(2,1),(4,2),(3,3),(5,4)$.
   (d) Find the Jacobian of G' and give a geometric explanation for the similarity between the Jacobian of G and that of G'.

2. Consider the region $\mathcal{D}$ defined by $1 \le x^2 - y^2 \le 4$ and $0 \le y \le \dfrac{3x}{5}$. In this problem you'll set up an integral to compute $\displaystyle\iint_{\mathcal{D}} e^{x^2-y^2}\, dA$.

   Consider the change of coordinates $G(u,v) = \left(\dfrac{v}{2} + \dfrac{u}{2v}, \dfrac{v}{2} - \dfrac{u}{2v}\right)$. Recall from class that the inverse of this coordinate change is given by $G^{-1}(x,y) = (x^2-y^2, x+y)$

   (a) Find a region $\mathcal{R}$ of the uv-plane so that $G : \mathcal{R} \to \mathcal{D}$ is a change of coordinates map (so G is onto and one-to-one on the interior of $\mathcal{R}$). **Hint:** Start by finding 4 curves in the uv-plane that map to the 4 curves forming the boundary of $\mathcal{D}$.
   (b) Give an iterated integral in uv-coordinatees to compute $\displaystyle\iint_{D} e^{x^2-y^2}\, dA$ (No need to compute the actual integral, but it is an integral you can compute).

3. Consider the region of the part of the first quadrant $\mathcal{D}$ defined by $1 \le x^2+y^2 \le 4$ and $1/10 \le xy \le 1/2$ and $y \ge x$. There is a change of coordinates G that takes the rectangle $[1,4]\times[1/10,1/2]$ in the uv-plane to $\mathcal{D}$, and the *inverse* of this change of coordinates is given by $G^{-1}(x,y) = (x^2+y^2, xy)$.

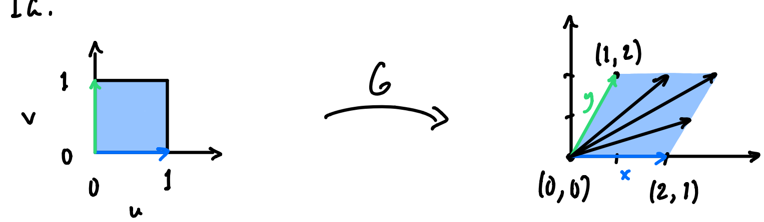


(a) What is the absolute value of the Jacobian of G^{-1}? (It should be a function of x and y). Pay attention to signs!

(b) Compute $\iint_{\mathcal{D}} y^2 - x^2 \, dA$

(c) Bonus problem: Note that the system of inequalites $1 \le x^2 + y^2 \le 4$ and $1/10 \le xy \le 1/2$ defines *four* different regions of the plane. Each of these regions can be described by a change of coordinates G that takes the rectangle $[1,4] \times [1/10, 1/2]$ in the uv-plane to the region where again *inverse* of this change of coordinates is given by $G^{-1}(x,y) = (x^2 + y^2, xy)$, but for each of these regions G itself has a different formula. Find all 4 formula for G and say which of these four regions goes with which formula.

1a.



$$G(u,v) = u\vec{x} + v\vec{y} = u(2,1) + v(1,2) = (\underline{2u+v}, \underline{u+2v})$$

b. $$J_G = \begin{vmatrix} \partial G_1/\partial u & \partial G_1/\partial v \\ \partial G_2/\partial u & \partial G_2/\partial v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

c. This is the same parallelogram, but shifted by $(2,1)$. So we use

$$\tilde{G}(u,v) = (2,1) + G(u,v) = (2 + 2u + v, \; 1 + u + 2v)$$

and we also have

d. $$J_{\tilde{G}} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

N.b. I don't recommend calling this map G', since the <u>matrix</u> $\begin{bmatrix} \partial G_1/\partial u & \partial G_1/\partial v \\ \partial G_2/\partial u & \partial G_2/\partial v \end{bmatrix}$ for G is usually called G'.

2a. $1 = \left(\frac{v}{2} + \frac{u}{2v}\right)^2 - \left(\frac{v}{2} - \frac{u}{2v}\right)^2 = \cancel{\frac{v^2}{4}} + \cancel{\frac{u^2}{4v^2}} + 2\frac{v}{2}\cdot\frac{u}{2v} - \cancel{\frac{v^2}{4}} - \cancel{\frac{u^2}{4v^2}} - -2\frac{v}{2}\frac{u}{2v}$

$= u$

so $\underline{u=1}$ corresp. to $x^2 - y^2 = 1$.

Likewise $\underline{u=4}$ corresp. to $x^2-y^2=4$.

$y=0$ corresponds to $\frac{v}{2} - \frac{u}{2v} = 0 \iff v^2 - u = 0 \iff u = v^2 \iff v = \sqrt{u}$

$y = \frac{3x}{5}$ corresp. to $\frac{v}{2} - \frac{u}{2v} = \frac{3}{5}\left(\frac{v}{2} + \frac{u}{2v}\right) \iff u = \frac{1}{4}v^2 \iff \overline{\underline{v = 2\sqrt{u}}}$

$\Omega = \{(u,v) : 1 \le u \le 4, \sqrt{u} \le v \le 2\sqrt{u}\}$

Note that $x, y \ge 0$ in this region, so $v = x + y \ge 0$. Hence we use the positive square root.

b. $\partial G_1/\partial u = \frac{1}{2v}$ $\quad \partial G_1/\partial v = \frac{1}{2} - \frac{u}{2}v^{-2}$

$\partial G_2/\partial u = -\frac{1}{2v}$ $\quad \partial G_2/\partial v = \frac{1}{2} + \frac{u}{2}v^{-2}$

so the Jacobian is

$$\det G' = \frac{1}{2v}\left(\frac{1}{2} + \cancel{\frac{u}{2}v^{-2}}\right) - -\frac{1}{2v}\left(\frac{1}{2} - \cancel{\frac{u}{2}v^{-2}}\right) = \frac{1}{2v}$$

$$\iint_D e^{x^2-y^2}\,dA = \iint_\Omega e^u |\det G'(u,v)|\,du\,dv = \int_1^4 \int_{\sqrt{u}}^{2\sqrt{u}} e^u \cdot \frac{1}{2v}\,dv\,du$$

$$= \frac{1}{2}e(e^3-1)\log 2$$

3a. $\det(G^{-1})' = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2(x^2 - y^2)$

Since $y \ge x$ in this region, $|\det(G^{-1})'| = 2|x^2-y^2| = 2(y^2-x^2)$.

b. Thus by the change of variables formula,

$$\int_D y^2 - x^2\,dx\,dy = \int_D \frac{1}{2}\underbrace{|\det(G^{-1})'|}_{2(y^2-x^2)}\,dx\,dy = \int_{G^{-1}(D)} \frac{1}{2}\,du\,dv = \frac{1}{2}\cdot(4-1)\cdot\left(1/2 - 1/10\right)$$

$G^{-1}(D) = [1,4] \times [1/10, 1/2]$

c. $u = x^2 + y^2$ solve for x, y :

$v = xy$

$$u - x^2 = y^2 = \frac{v^2}{x^2}$$

$$x^2 u - (x^2)^2 = v^2$$

$$(x^2)^2 - x^2 u + v^2 = 0$$

$$x^2 = \frac{u \pm \sqrt{u^2 - 4v^2}}{2}$$

$$x = \pm \sqrt{\frac{u \pm \sqrt{u^2 - 4v^2}}{2}}, \quad y = \pm \sqrt{\frac{v^2}{x^2}} \text{ in terms of } x$$

$$\left(= \pm \frac{2|v|}{\sqrt{|u \pm \sqrt{u^2 - 4v^2}|}} \right)$$

choosing the signs:

for I: $+, -, +$

II: $+, +, +$

III: $-, +, -$

IV: $-, -, -$