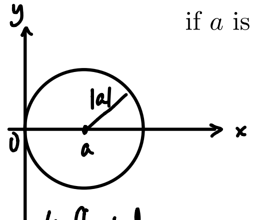
1. We showed that the circle of radius a has the concise description in polar coordinates as $r = a$. Find a description of the circles $(x-a)^2 + y^2 = a^2$ and $x^2 + (y-a)^2 = a^2$ in polar coordinates. (What if a is negative?)



$x = r\cos\theta,\ y = r\sin\theta$

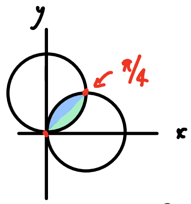
$\Rightarrow\ (r\cos\theta - a)^2 + r^2\sin^2\theta = a^2$

$r^2\cos^2\theta - 2ar\cos\theta + r^2\sin^2\theta = 0$

$r = 2a\cos\theta$ & sim. for other one

(reflected over y-axis if $a<0$)

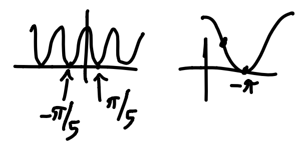
2. Compute the area inside the curves $r = \cos\theta$ and $r = \sin\theta$.

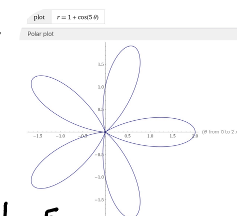


Using a half-angle formula $\left(\frac{1+\cos 2\theta}{2} = \cos^2\theta\right)$, $\int \cos^2\theta\, d\theta = \left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right) + \text{const.}$

$$A = \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos^2\theta\, d\theta + \frac{1}{2}\int_0^{\pi/4} \sin^2\theta\, d\theta = 2\,\frac{\pi-2}{16} = \frac{\pi-2}{8}$$

3. (a) Plot the curve $r = 1 + cos(5\theta)$ (it should look something like a flower).
   (b) Find the area of one "petal" of the curve $r = 1 + \cos 5\theta$. You will probably need to use the double angle formula.





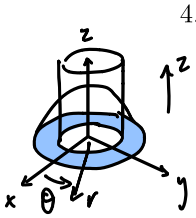
(b) Look for two consecutive zeros of $1 + \cos 5\theta$:

$$A = \frac{1}{2}\int r^2\, d\theta = \frac{1}{2}\int_{-\pi/5}^{\pi/5} (1+\cos 5\theta)^2\, d\theta$$

$\hookrightarrow 1 + 2\cos 5\theta + \cos^2 5\theta$

$$= \frac{1}{2}\left(\frac{3\theta}{2} + \frac{2}{5}\sin 5\theta + \frac{1}{20}\sin 10\theta\right)\Big|_{-\pi/5}^{\pi/5} = \frac{3\pi}{10}$$ or find full area & divide by 5

4. If a solid in a region $\mathcal{W} \subset \mathbb{R}^3$ has density given by $\partial(x,y,z)$ then its mass is given by $\iiint_{\mathcal{W}} \partial(x,y,z)\, dV$.



Suppose that we are measuring in meters and consider a solid that is above the plane $z = 0$, below the paraboloid $z = 4 - (x^2+y^2)$, and outside the cylinder $x^2 + y^2 = 1$. Suppose that the density of this solid is inversely proportional to the distance from the z-axis and that the density of this solid along the boundary where the paraboloid hits the xy-plane is $1/2\ kg/m^3$.

$\partial(x,y,z) = \|(x,y,z) - (0,0,z)\|^{-1} = (x^2+y^2)^{-1/2} = \frac{1}{r}$

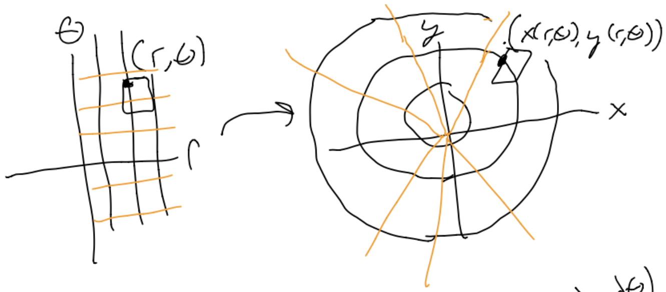
Compute the mass of this solid.

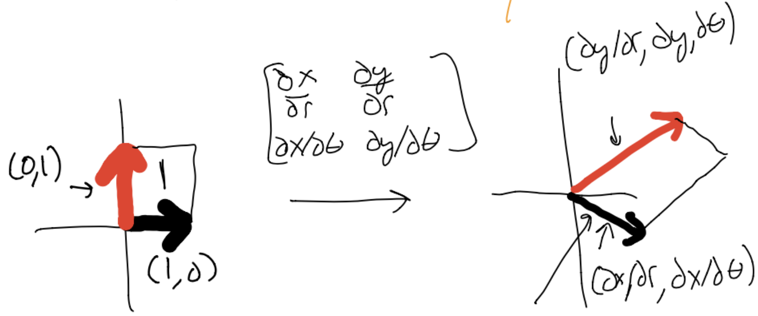
In cylindrical coords.,

$$m = \int_0^{2\pi}\int_1^2\int_0^{4-r^2} \left(\frac{1}{r}\right) \underbrace{r\, dz\, dr\, d\theta}_{dz\,(r\,dr\,d\theta)} = 2\pi\int_1^2 (4-r^2)\, dr = 2\pi\cdot\frac{5}{3} = \frac{10\pi}{3}$$

5. In this problem you will find the area of the ellipse $(x/a)^2 + (y/b)^2 = 1$. We'll use a distorted version of polar coordinates. We'll measure points in the plane by the angle θ the line from the origin to the point makes with the x-axis and the value of r for which the point lies on the ellipse $\left(\frac{x}{ar}\right)^2 + \left(\frac{y}{br}\right)^2 = 1$.
   (a) Using these coordinates what point in the xy-plane does the value (r,θ) correspond to? $(ar\cos\theta, br\sin\theta)$
   (b) Describe the ellipse $(x/a)^2 + (y/b)^2 = 1$ in these new "distorted polar coordinates". $r = 1$
   (c) What is the distortion factor for area with these coordinates?
   For this, consider the map $(r,\theta) \mapsto (x(r,\theta), y(r,\theta))$ from the first part of the question. Differentiating this gives the matrix $\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix} =: J_G$, and the area distortion factor at (r,θ) is the absolute value of the determinant of this

matrix, the quantity $\dfrac{\partial x}{\partial r}\dfrac{\partial y}{\partial \theta} - \dfrac{\partial y}{\partial r}\dfrac{\partial x}{\partial \theta}$. See the picture below.





the area of the square $[0,1]\times[0,1]$ under the this map is
$\partial x/\partial r \cdot \partial y/\partial\theta - \dfrac{\partial y}{\partial r}\cdot\dfrac{\partial x}{\partial \theta}$

(d) What is the area enclosed by this ellipse?

(c) $x = ar\cos\theta,\ y = br\sin\theta$

$$\frac{\partial x}{\partial r} = a\cos\theta \qquad \frac{\partial y}{\partial r} = b\sin\theta$$

$$\frac{\partial x}{\partial \theta} = -ar\sin\theta \qquad \frac{\partial y}{\partial \theta} = br\cos\theta$$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = abr\cos^2\theta - (-abr\sin^2\theta) = abr$$

so

(d) writing $G(r,\theta) = (ar\cos\theta, br\sin\theta)$, the ellipse is the image of $\overbrace{[0,1]}^{r} \times \overbrace{[0,2\pi]}^{\theta}$ under G, so

$$A(\text{ellipse}) = \int_{\text{ellipse}} 1 \, dx\, dy = \int_{G([0,1]\times[0,2\pi])} dx\, dy$$

by the change of variables formula, more on this later

$$= \int_{[0,1]\times[0,2\pi]} |\det J_G| \, dr\, d\theta = 2\pi \int_0^1 abr\, dr = \pi ab$$