WORKSHEET
DISCUSSION SECTION 2
DUE 4/15 AT MIDNIGHT PDT

(1) Consider the following situation. Alice sets up coordinates on $\mathbb{R}^3$ (3D space). Next, Bob sets up his own coordinates by:
  (i) putting his origin at the point Alice calls $(2, 3, 4)$,
  (ii) taking his x-axis parallel to the line $y = x$ in Alice's xy plane and pointing towards Alice's first octant,
  (iii) taking his z-axis in the same direction as Alice's z-axis,
  (iv) using a unit that is Alice's unit, but scaled up by a factor of $\sqrt{2}$.

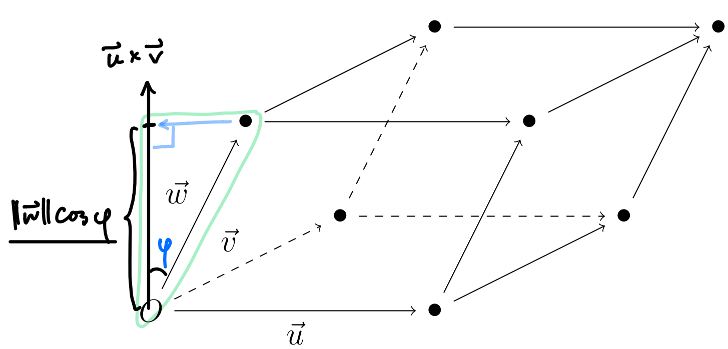
  Let P be the point Bob calls $(1, 1, \sqrt{2})$. What does Alice call this point?

(2) Consider the vectors

$$\vec{u} = \langle 1, 2, 3\rangle \quad , \quad \vec{v} = \langle 4, 5, 6\rangle \quad , \quad \vec{w} = \langle 7, 8, 10\rangle.$$

  (a) Find scalars a, b, c such that $a\vec{i} + b\vec{j} + c\vec{k} = \langle 13, 17, 23\rangle$.
  (b) Find scalars d, e, f such that $d\vec{u} + e\vec{v} + f\vec{w} = \langle 13, 17, 23\rangle$.
  (c) What is the orthogonal projection of $\langle 13, 17, 23\rangle$ onto $\vec{i}$? Is it equal to the vector $a\vec{i}$ from part (a)?
  (d) What is the orthogonal projection of $\langle 13, 17, 23\rangle$ onto $\vec{u}$? Is it equal to the vector $d\vec{u}$ from part (b)?
  (e) Why are your answers in (c) and (d) different? Hint: compute the angles between $\vec{u}$, $\vec{v}$, and $\vec{w}$.

(3) Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$. Explain why $|(\vec{u} \times \vec{v}) \bullet \vec{w}|$ is the volume of the 3-parallelepiped (3D parallelogram) below.

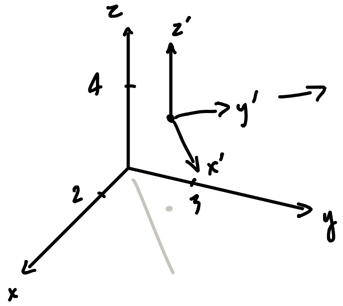


$$\text{vol} = \underbrace{\|\vec{u} \times \vec{v}\|}_{\text{area of parallellogram}} \cdot \underbrace{\|\vec{w}\| |\cos \varphi|}_{\text{height of parallelepiped}}$$

$$= |(\vec{u} \times \vec{v}) \cdot \vec{w}|$$

Think about shearing a deck of cards.

φ = angle between $\vec{w}$ and vertical
= angle between $\vec{w}$ and $\vec{u} \times \vec{v}$

1. 

P in Alice's coords.

$$= (2,3,4) + \sqrt{2}\cdot 1 \cdot \frac{(1,1,0)}{\sqrt{2}} + \sqrt{2}\cdot 1\,\frac{(-1,1,0)}{\sqrt{2}} + \sqrt{2}\cdot\sqrt{2}\,(0,0,1)$$

$$= (2,3,4) + (\cancel{1},1,0) + (\cancel{-1},1,0) + (0,0,2)$$

$$= \underline{(2,5,6)}$$

2. $\vec{u} = (1,2,3)$, $\vec{v} = (4,5,6)$, $\vec{w} = (7,8,10)$

(a) $a\hat{\imath} + b\hat{\jmath} + c\hat{k} = (13,17,23) \Rightarrow a, b, c = 13, 17, 23$

(b) $d\vec{u} + e\vec{v} + f\vec{w} = (13,17,23)$

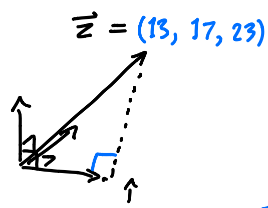
$$\Rightarrow (d + 4e + 7f,\ 2d + 5e + 8f,\ 3d + 6e + 10f) = (13,17,23)$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 13 \\ 2 & 5 & 8 & 17 \\ 3 & 6 & 10 & 23 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 7 & 13 \\ 0 & -3 & -6 & -9 \\ 0 & -6 & -11 & -16 \end{array}\right]$$

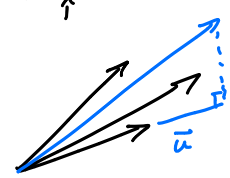
$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 7 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\Rightarrow f = 2,\quad e + 2\cdot 2 = 3,\quad d + 4(3-4) + 7\cdot 2 = 13$$

$$\Rightarrow d = 3,\ e = -1,\ f = 2$$



(c) $$\mathrm{proj}_{\hat{\imath}}(13,17,23) = \frac{(13,17,23)\cdot\hat{\imath}}{\underset{\to 1}{\cancel{\|\hat{\imath}\|^2}}}\,\hat{\imath} = \underline{13\hat{\imath}} = a\hat{\imath}$$



(d) $$\mathrm{proj}_{\hat{u}}(13,17,23) = \frac{(13,17,23)\cdot(1,2,3)}{1^2+2^2+3^2}\,\hat{u} = \frac{58}{7}\vec{u} \neq d\vec{u}$$

(e)
$$\left.\begin{aligned} \vec{u}\cdot\vec{v} &= (1,2,3)\cdot(4,5,6) > 0 \\ \vec{v}\cdot\vec{w} &= (4,5,6)\cdot(7,8,10) > 0 \\ \vec{u}\cdot\vec{w} &= (1,2,3)\cdot(7,8,10) > 0 \end{aligned}\right\}$$

so $\vec{u}, \vec{v}, \vec{w}$ are not orthogonal (in fact, no two are orthogonal).

3. $$|(\vec{u}\times\vec{v})\cdot\vec{w}| = \underbrace{\|\vec{u}\times\vec{v}\|}_{\text{area of parallelogram}}\ \underbrace{\|\vec{w}\|\cos\varphi}_{\text{height of parallelepiped}} = \text{vol. of parallelepiped}$$

(see diagram)