disc. 1

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Operations on sets

Union $\bigcup_{\alpha \in \mathcal{A}} B_\alpha = \{x : \text{there exists } \alpha \text{ s.t. } x \in B_\alpha\}$ "or"

Intersection $\bigcap_{\alpha \in \mathcal{A}} B_\alpha = \{x : x \in B_\alpha \text{ for every } \alpha\}$ "and"

Complement $A^c = \{x \in \Omega : x \in A\}$ "not"

Difference $A \setminus B = \{x \in A : x \notin B\}$

$A = B$ means: $x \in A \Leftrightarrow x \in B$.

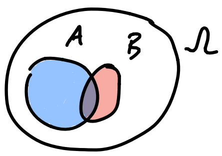
$A \subseteq B$ means: $x \in A \Rightarrow x \in B$

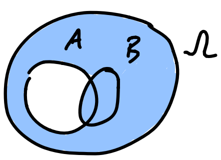
Useful properties:

$A \setminus B = A \cap B^c$

$$\left.\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \end{aligned}\right\} \text{distributivity}$$

Ex. $\Omega \setminus (A \setminus B) = (\Omega \setminus A) \cup B$





Pf. $x \in \Omega \setminus (A \setminus B)$

iff $x \in \Omega$ and $x \notin A \setminus B$

iff $x \in \Omega$ and $x \notin A \cap B^c$

iff $\neg(x \in A \text{ and } x \in B^c)$

iff $\neg(x \in A \text{ and } x \notin B)$

iff $x \notin A$ or $x \in B$

iff $x \in \Omega \setminus A$ or $x \in B$

iff $x \in (\Omega \setminus A) \cup B$.

Probability spaces

$(\Omega, \mathcal{F}, P)$

Ω → sample space

$\mathcal{F}$ → event space (σ-algebra)

P → probability measure

σ-algebra:

$\emptyset \in \mathcal{F}$

$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

$A_1, A_2, \ldots \in \mathcal{F} \Rightarrow A_1 \cup A_2 \cup \cdots \in \mathcal{F}$

probability measure:

$$\begin{cases} P(\cdot) \geq 0 \\ P(\Omega) = 1, \quad P(\emptyset) = 0 \\ A_i \in \mathcal{F}, \quad A_i \cap A_j = \emptyset \text{ for } i \neq j \\ \quad \Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \end{cases}$$

Exm. 1.23 (Uniform dist on N elements)

If $\Omega = \{\omega_1, \dots, \omega_N\}$, $\mathcal{F} = 2^{\Omega}$, $P(\omega_i) = P(\omega_j)\ \forall i \forall j$,
then P is a prob. measure if $P(\omega_i) = 1/N$.
Then $P(A) = \frac{|A|}{N}$.

N.b. Can't have a uniform dist. on infinitely many elements.

Ex. 1.22a

Consider an experiment in which we roll a fair coin (H/T) 10 times.
Describe the prob. space.

Ω = finite sequences of length 10 on $\{H, T\}$
$\mathcal{F} = 2^{\Omega}$
P = uniform prob. measure

(b)
ω_0 = 0 tails (H H H ...)
ω_1 = 1 tail (T H H ...), (H T H ...), etc.
ω_2 = 2 tails ...
...
ω_{10} = 10 tails

$\Omega = \{\omega_0, \dots, \omega_{10}\}$

Ex. 1.59 Find the probability of n heads in $2n$ tosses of a fair coin.

$$\binom{2n}{n} \frac{1}{2^{2n}}$$

disc. 2

From hwk 1

#7a Suppose $P(A) = 3/4$, $P(B) = 1/3$. Show that

$$\frac{1}{12} \le P(A \cap B) \le \frac{1}{3}$$

and show the extreme values are attained.

Pf. (upper) By monotonicity, $A \cap B \subseteq B \Rightarrow P(A \cap B) \le P(B) = 1/3$.

(lower) We have $P(A \cup B) \le 1$. By inclusion-exclusion,

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 3/4 + 1/3 - P(A \cup B) \\ &\ge 3/4 + 1/3 - 1 = 1/12. \end{aligned}$$

Exm. $\Omega = [0, 1]$, $\mathcal{F} = $ Borel, $P = $ Lebesgue meas.

- $A = [0, 3/4]$, $B = [2/3, 1] \Rightarrow A \cap B = [2/3, 3/4]$

  $\Rightarrow P(A) = 3/4$, $P(B) = 1/3$, $P(A \cap B) = 1/12$

- $A = [0, 3/4]$, $B = [0, 1/3] \Rightarrow A \cap B = [0, 1/3]$

  $\Rightarrow P(A) = 3/4$, $P(B) = 1/3$, $P(A \cap B) = 1/3$

#3 (countable version) Suppose $\Omega = \{\omega_1, \omega_2, \ldots\}$ is countable, $\sum_{i=1}^{\infty} p_i = 1$, $\mathcal{F} = 2^{\Omega}$.

We set $P(\{\omega_i\}) = p_i$, so that

$$P(A) = \sum_{i : \omega_i \in A} p_i.$$

Clearly $P(\Omega) = 1$.

If $A_1, A_2, \ldots$ are disjoint, then

use disjointness

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{i : \omega_i \in \bigcup_j A_j} p_i = \sum_{j=1}^{\infty} \sum_{i : \omega_i \in A_j} p_i = \sum_{j=1}^{\infty} P(A_j)$$

Examples Fair die: $\Omega = \{1, 2, \ldots, 6\}$, $p_1 = \cdots = p_6 = 1/6$

Urn w/ 3 red balls, 2 blue balls, 1 green ball

$\Omega = \{R, G, B\}$, $p_R = 3/6$, $p_B = 2/6$, $p_G = 1/6$

Bayes' rule & example

$$P(A \mid B) = \frac{P(A)}{P(B)} P(B \mid A) \qquad (P(A),\ P(B) > 0)$$

Ex. $A = \{\text{bomber is there}\}$

$B = \{\text{radar alerts you}\}$

$P(A) = 0.05$

$P(B \mid A) = 0.99$ (detection)

$P(B \mid A^c) = 0.1$ (false alarm)

If the alarm goes off, what is the probability that the bomber is actually there?

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B)} = \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(A^c)P(B \mid A^c)} \approx 0.343$$

With $P(A) = 0.05$

$P(B \mid A) = 0.995$ get $P(A \mid B) \approx 0.567$

$P(B \mid A^c) = 0.04$,

Monty Hall problem You are presented with 3 doors.

- Behind one is a car, behind the others are goats.
- You select a door.
- The host opens, at random, a door that you did not pick and that does not have the car behind it.

Q. Should you switch to the other door?

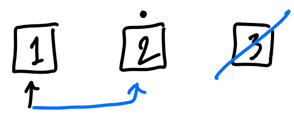
If you don't switch, $P(\text{win}) = 1/3$

If you switch,

w/ prob. $1/3$:

lose

w/ prob. $2/3$:



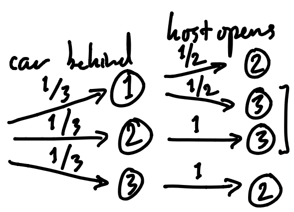
win

$\Rightarrow P(\text{win}) = 2/3$.

More formally, suppose we choose door 1 and will switch.

$P(\text{win} \mid \text{host opens door } 3) = P(\text{car at } 2 \mid \text{host opens door } 3)$

$= P(\text{car at } 2\ \&\ \text{host opens door } 3) / P(\text{host opens door } 3)$



$\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{1}{2}$

$= \dfrac{1/3}{1/2} = \dfrac{2}{3}$.

Note: Pigeons are good at this problem. [Herbranson & Schroeder 2010]

"Replication of the procedure with human participants showed that humans failed to adopt optimal strategies, even with extensive training."

disc. 3

Review of a few problems

1(a). $\binom{n}{k}$ ways of getting k heads in n tosses:

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!} \quad \text{sequences with } k \text{ heads}$$

But we just want strings with k heads irresp. of order, so divide by $k!$ to get

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

3. (first method) Show $A_1^c, A_2, \dots, A_n$ are indep.; then done by iterating this.

Fix a subcoll. $\{i_1, \dots, i_k\} = J \ (1 \le i_j \le n)$

If $1 \notin J$, then done.

Otherwise, write $J = \{1\} \dot\cup J_0$. Then (disj. union)

$$\bigcap_{j \in J_0} A_j = \left(\left(\bigcap_{j \in J_0} A_j\right) \cap A_1\right) \dot\cup \left(\left(\bigcap_{j \in J_0} A_j\right) \cap A_1^c\right)$$

so

$$\underbrace{P\left(\bigcap_{j \in J_0} A_j\right)}_{=\prod_{j \in J_0} P(A_j)} = \underbrace{P\left(\bigcap_{j \in J} A_j\right)}_{=\prod_{j \in J} P(A_j)} + P\left(A_1^c \cap \bigcap_{j \in J_0} A_j\right)$$

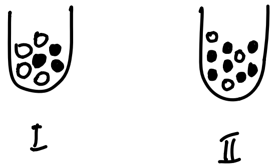
$$\Rightarrow \quad P\left(A_1^c \cap \bigcap_{j \in J_0} A_j\right) = \prod_{j \in J_0} P(A_j) - \prod_{j \in J_0} P(A_j) P(A_1) = (1 - P(A_1)) \prod_{j \in J_0} P(A_j)$$

as desired.

(second method)

$$\begin{aligned}
P(A_{i_1}^c \cap \cdots \cap A_{i_k}^c) &= 1 - P(A_{i_1} \cup \dots \cup A_{i_k}) \\
&= 1 - \sum_{\emptyset \ne J \subseteq \{i_1, \dots, i_k\}} (-1)^{|J|+1} P\left(\bigcap_{j \in J} A_j\right) \quad \text{(inclusion-exclusion)} \\
&= 1 - \sum_{''} (-1)^{|J|+1} \prod_{j \in J} P(A_j) \\
&= \prod_{j=1}^{k} (1 - P(A_{i_j})) \quad \text{(by induction)}
\end{aligned}$$

6. a.



$$P(B) = P(B \mid I)\,P(I) + P(B \mid II)\,P(II)$$

$$= \frac{3}{7} \cdot \frac{1}{2} + \frac{7}{10} \cdot \frac{1}{2} = 79/140 \approx 0.564$$

b. $$P(I \mid W) = P(W \mid I)\,\frac{P(I)}{P(W)}$$

$$= \frac{4}{7} \cdot \frac{1/2}{1 - P(B)} = 40/61 \approx 0.655$$

Thm (Cantor) $|S| < |2^S|$

Pf. $x \mapsto \{x\}$ is an injection : $S \to 2^S$, so $|S| \leq |2^S|$. Suppose $f : S \to 2^S$ is surjective. Consider $E = \{\xi \in S : \xi \notin f(\xi)\}$. Then $\exists \xi : f(\xi) = E$. But then

$\xi \in E$ iff $\xi \notin f(\xi) = E$, a contradiction. So $|S| < |2^S|$.

Cor. There is no set of all sets.

Pf. If S were a set of all sets, then $2^S \in S$, so $|2^S| \leq |S|$.
By Cantor's thm., $|2^S| \leq |S| < |2^S|$, a contradiction.

disc. 4

Review prob. 6

6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\Omega = \{1,2,3,4,5,6\}$ and $\mathcal{F} = \{\emptyset, \{2,4,6\}, \{1,3,5\}, \Omega\}$. Let U, V, W be mappings $\Omega \to \mathbb{R}$ defined by

$$U(\omega) = \omega, \qquad V(\omega) = \begin{cases} 1 \text{ if } \omega \text{ is odd,} \\ 0 \text{ if } \omega \text{ is even} \end{cases}, \qquad W(\omega) = \omega^2,$$

for $\omega \in \Omega$. Determine which of U, V, W are discrete random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. In each case justify your answer.

V is a discrete RV, but U, W aren't, because they're not measurable w.r.t. $\mathcal{F}$.

Ex. 2.18

Toss a coin n times, w/ prob. p of heads on each toss

$$\Omega = \{ s_1 \cdots s_n : s_i \in \{H, T\}\}, \quad \mathcal{F} = 2^{\Omega},$$

$$P(\{\omega\}) = p^{h(\omega)} q^{n-h(\omega)}$$

where $h(\omega)$ is the number of heads and $q = 1-p$.

Set

$$X_i(\omega) = \begin{cases} 1 & \text{if the } i^{th} \text{ entry of } \omega \text{ is } H \\ 0 & \text{" ——— is } T \end{cases}$$

Then

$$\begin{aligned} P(X_i = 0) &= P(\{\omega \in \Omega : \omega_i = T\}) \\ &= \sum_{\omega : \omega_i = T} P(\{\omega\}) \\ &= \sum_{h=0}^{n-1} \sum_{\substack{\omega : \omega_i = T \\ h(\omega) = h}} p^h q^{n-h} = q \sum_{h=0}^{n-1} \binom{n-1}{h} p^h q^{n-1-h} \\ &= q(p+q)^{n-1} = q \end{aligned}$$

so $P(X_i = 1) = p$

$\Rightarrow X_i \sim \text{Bernoulli}(p)$.

Now set

$$S_n = X_1 + X_2 + \cdots + X_n,$$

the total number of heads.

We have

$$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\underset{\lambda = np}{=} \frac{n(n-1)\cdots(n-k+1)}{k!} \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \underbrace{\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-k+1}{n}}_{\longrightarrow 1} \frac{\lambda^k}{k!} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\longrightarrow e^{-\lambda}} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-k}}_{\longrightarrow 1}$$

as $n \to \infty$

$$\longrightarrow \frac{\lambda^k}{k!} e^{-\lambda}$$

so we obtain a Poisson variable X w/

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad (k \geq 0)$$

<u>Ex.</u> Guardian newspaper w/ 10^6 chars., prob. 10^{-5} of an error

Get $\lambda = 10$ so e.g.

$$P(10 \text{ errors}) = P(S_n = 10) \approx \frac{1}{10!} e^{-10} \cdot 10^{10} \approx 0.125$$

<u>Bulmer horse example</u>

Example from von Bortkiewicz <u>Das Gesetz der kleinen Zahlen</u> (1898)
related in Bulmer <u>Principles of Statistics</u>

TABLE 13

The annual numbers of deaths from horse kicks in 14 Prussian army corps between 1875 and 1894

Number of deaths	Observed frequency	Expected frequency
0	144	139
1	91	97
2	32	34
3	11	8
4	2	1
5 and over	0	0
Total	280	280

Von Bortkiewicz extracted from official records the numbers of deaths from horse kicks in 14 army corps over the twenty-year period 1875-1894, obtaining 280 observations in all. He argued that the chance that a particular soldier should be killed by a

$$\lambda = \frac{196}{280} = 0.7$$

(i.e. using the empirical mean)

$280 \cdot e^{-0.7}$ etc.

disc. 5

Hwk. problems

5. $X \sim \text{Binom}(0.2)$ number of wrong decisions

$$P(\text{wrong}) = P(X \geq 4) = \sum_{k=4}^{7} \binom{7}{k} 0.2^k \, 0.8^{7-k}$$

7. $N \sim \text{Geom}(0.05)$

$$P(N=k) = (1-p)^{k-1} p$$

$$\frac{1-0.9^{k+1}}{0.1} \cdot 1000$$

$$y(k) = 1000 \cdot 0.9^k$$

$$p(k) = \sum_{j=1}^{k} 1000 \cdot 0.9^j$$

$$P\left(p(N) = \sum_{j=1}^{k} 1000 \cdot 0.9^j\right) = (1-p)^{k-1} p$$

0 otherwise

$$E[p(N)] = \sum_{k=1}^{\infty} \sum_{j=0}^{k} 1000 \cdot 0.9^j (1-p)^{k-1} p$$

$$= 500 \sum_{k=1}^{\infty} (1 - 0.9^{k+1}) \, 0.95^{k-1}$$

$$= 7206.9$$

another geom. example?

Exr. Suppose you flip a fair coin repeatedly until you see a heads followed by a tails. What is the expected number of coin flips?

$$\underbrace{TTT \cdots TH}_{\text{geom}(1/2)}\underbrace{H \cdots HT}_{\text{geom}(1/2)} \longrightarrow E[\text{geom}(1/2) + \text{geom}(1/2)] = 4$$

more generally $\text{geom}(p)$ $\text{geom}(q)$ $\longrightarrow E[\text{geom}(p) + \text{geom}(q)] = \frac{1}{p} + \frac{1}{q}$

Exponential RV/ memoryless prop.

We say T is exponential if $P(T > t) = e^{-\lambda t}$ for some $\lambda > 0$.

Can check T is memoryless directly.

Converse (Durrett):

$P(T > 0) = 1$ and $P(T > t + s \mid T > t) = P(T > s)$ $\quad (t, s \geq 0)$

$\Rightarrow P(T > t) = e^{-\lambda t}$ for $t \geq 0$ $\quad$ (first do it for $t = m 2^{-n}$)

i.e. $\quad P(T > m + k) = P(T > k) P(T > m)$

$$f(t) = P(T > t)$$

$$f(x + y) = f(x) f(y)$$

$\Rightarrow f(0) = 1$ since $f(0) > 0$.

Also $\Rightarrow f(m) = f(1)^m$ & $f\left(\frac{1}{n}\right)^n = f(1)$

$$\Rightarrow f\left(\tfrac{1}{n}\right) = f(1)^{1/n}$$

so $f\left(\frac{p}{q}\right) = f\left(\frac{1}{q}\right)^p = f(1)^{p/q}$ so def. λ s.t. $f(1) = e^{-\lambda}$ (since $f(1) \neq 0$)

$\Rightarrow f(r) = e^{-\lambda r}$ for $r \in \mathbb{Q}^+$. Then use monotonicity:

note that $s < t \Rightarrow \{T > s\} \supseteq \{T > t\}$ so $P(T > \cdot)$ is decreasing.

So for $t > 0$, if q_-, q_+ are rational and $q_- < t < q_+$, we have

$$\begin{array}{ccccc} P(T > q_-) & \geq & P(T > t) & \geq & P(T > q_+) \\ \| & & & & \| \\ e^{-\lambda q_-} & \geq & e^{-\lambda t} & \geq & e^{-\lambda q_+} \end{array}$$

so sending $q_- \nearrow t$, $q_+ \searrow t$, by continuity of $e^{-\lambda t}$ we get

$$P(T > t) = e^{-\lambda t} \quad \forall t \geq 0.$$

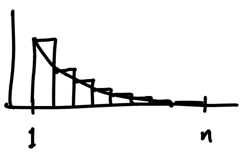
disc. 6

3(c) From (b),

$$\mathbb{E}[X] = \mathbb{E}\left[\text{geom}(1) + \text{geom}\left(\tfrac{m-1}{m}\right) + \cdots + \text{geom}\left(\tfrac{1}{m}\right)\right]$$

$$= \frac{m}{m} + \frac{m}{m-1} + \cdots + \frac{m}{1} = m\left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right) = m H_m$$

We have $c_n = \sum_{k=1}^{n} \frac{1}{k} - \underbrace{\log n}_{\int_1^n \frac{1}{y}\,dy} \longrightarrow \gamma = 0.57721\ldots$



$$\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{y}\,dy \geq 0$$

claim it's decr., $c_{n+1} \leq c_n$

$$\left(\sum_{k=1}^{n+1} \frac{1}{k} - \log(n+1)\right) - \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) = \frac{1}{n+1} - \log\left(\frac{n+1}{n}\right)$$

$$= \left(\frac{1}{n+1} - \frac{1}{n}\right) + \frac{1}{n} - \log\left(1 + \frac{1}{n}\right)$$

$|x| < 1$ $\quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$$x - \log(1+x) = \frac{x^2}{2} - \frac{x^3}{3} + \cdots \leq \frac{x^2}{2}$$

$$\leq \frac{-1}{n(n+1)} + \frac{1}{2n^2}$$

$$= \frac{-2n + (n+1)}{2n^2(n+1)} = \frac{-n+1}{2n^2(n+1)} \leq 0$$

$n H_n \approx n(\gamma + \log n) \approx 28.8$

↳ actually ≈ 29.3

6

4 aces / 52

13 spades

1	A	S
12	A^c	S
3	A	S^c
36	A^c	S^c

Y (spades)

X (aces)	0	1	2
0	$\frac{36\cdot 35}{52\cdot 51}$	$\frac{12\cdot 36}{52\cdot 51}\cdot 2$	$\frac{12\cdot 11}{52\cdot 51}$
1	$\frac{3\cdot 36}{52\cdot 51}\cdot 2$	$\left(\frac{1\cdot 36 + 12\cdot 3}{52\cdot 51}\right)\cdot 2$	$\frac{1\cdot 12}{52\cdot 51}\cdot 2$
2	$\frac{3\cdot 2}{52\cdot 51}$	$\frac{1\cdot 3}{52\cdot 51}\cdot 2$	0

$P(X=0, Y=0)$: 1^{st} draw in $A^c \cap S^c$ $\quad 36/52$

2^{nd} draw in $A^c \cap S^c$ $\quad 35/51$

$P(X=2, Y=2)$: 0

$P(X=0, Y=1)$: 1^{st} draw in $A^c \cap S$, 2^{nd} in $A^c \cap S^c$, or the other way around

$P(X=1, Y=1)$: 1^{st} draw in $A^c \cap S$, 2^{nd} in $A \cap S^c$ or other way around

OR 1^{st} draw in $A \cap S$, 2^{nd} in $A^c \cap S^c$ or other way around

7. X, Y RVs w/ $P(X = i, Y = j) = \theta^{i+j+1}$ $\quad 0 \le i, j \le 2$

$$\mathbb{E}[XY] = \sum_{i,j} ij\, P(X = i, Y = j) = 1 \cdot 1 \cdot \theta^3 + 1 \cdot 2 \cdot \theta^4 + 2 \cdot 1 \cdot \theta^4 + 2 \cdot 2 \cdot \theta^5$$

$$\begin{array}{ccc} \cancel{00} & \cancel{01} & \cancel{02} \\ \cancel{10} & 11 & 12 \\ \cancel{20} & 21 & 22 \end{array}$$

N.b.
$$\begin{aligned} & \theta + \theta^2 + \theta^3 \\ + & \theta^2 + \theta^3 + \theta^4 \\ + & \theta^3 + \theta^4 + \theta^5 \end{aligned}$$

$$\theta + 2\theta^2 + 3\theta^3 + 2\theta^4 + \theta^5 = 1$$

$\Rightarrow \theta \approx 0.40562$

$$P(X = i) = \sum_{j=0}^{2} P(X = i, Y = j) = \theta^{i+1} + \theta^{i+2} + \theta^{i+3}$$

$$\mathbb{E}[X] = \sum_{i=0}^{2} i\, P(X = i) = 1 \cdot (\theta^2 + \theta^3 + \theta^4) + 2 \cdot (\theta^3 + \theta^4 + \theta^5)$$

<u>Example</u> Every $x \in [0, 1)$ has a decimal expansion

$$x = 0.\varphi_1 \varphi_2 \varphi_3 \cdots \qquad \varphi_i : [0, 1) \to \{0, 1, 2, \dots, 9\}$$

<u>Claim:</u> the digits (i.e. the fns. φ_i) are independent

What we mean: the events $\underbrace{\{\varphi_1 = \alpha_1\}}_{\text{"1st digit is } \alpha_1\text{"}}, \{\varphi_2 = \alpha_2\}, \dots$ are independent

Want to show: $P(\varphi_{i_1} = \alpha_{i_1}, \dots, \varphi_{i_m} = \alpha_{i_m}) = P(\varphi_{i_1} = \alpha_{i_1}) \cdots P(\varphi_{i_m} = \alpha_{i_m})$.

e.g. $P(\underbrace{\text{1st digit is } 1}) = \frac{1}{10}$

$= \{x \in [0, 1) : 0.1 \le x < 0.2\}$

$P(\underbrace{\text{2nd digit is } 1}) = 10 \cdot \frac{1}{100} = \frac{1}{10}$

$= \{x : 0.{*}1 \le x < 0.{*}2$ for any $* \in \{0, \dots, 9\}\}$

& in general, $P(\text{digit } i_j = \alpha_j) = \frac{1}{10}$ for same reason.

Thus RHS is 10^{-m}. For LHS, such x are

$$x = 0.{*} \cdots {*}\alpha_{i_1}{*} \cdots {*}\alpha_{i_2}{*} \cdots \cdots {*}\alpha_{i_m}{*} \cdots\cdots$$

dec. place i_1, i_2, i_m

for each choice of stars, ~~the width~~ set of such x will have len. 10^{-i_m}.

Thus LHS = number of possible choices $\cdot\ 10^{-i_m}$.

How many stars? $i_m - 1 - (m - 1) = i_m - m$.

Therefore, $10^{i_m - m}$ choices, so LHS $= 10^{i_m - m} 10^{-i_m} = 10^{-m}$,

so the events are independent.

disc 7

5. A number N of balls is thrown at random into M boxes, with multiple occupancy allowed. Show that the expected number of empty boxes is $(M-1)^N/M^{N-1}$ and compute the variance of the number of empty boxes.
(Hint: Consider random variables $X_i = \mathbb{1}_{A_i}$ where $A_i =$ "the ith box is empty", and their sum.)

$X_i = 1_{A_i}$, $A_i = i^{th}$ box is empty N balls → M boxes

$X_1 + \cdots + X_M$ = # empty boxes

$$\mathbb{E}[X_1 + \cdots + X_M] = \sum_{i=1}^{M} \underbrace{P(\text{box } i \text{ is empty})}_{\left(\frac{M-1}{M}\right)^N}$$

$$= M\left(\frac{M-1}{M}\right)^N = \frac{(M-1)^N}{M^{N-1}}$$

$$\mathbb{E}[(X_1 + \cdots + X_M)^2] = \sum_{i,j=1}^{M} \mathbb{E}[X_i^2 X_j^2] = \sum_{i,j=1}^{M} \mathbb{E}[1_{A_i} 1_{A_j}] = \sum_{i,j=1}^{M} P(A_i \cap A_j)$$

$$= \sum_{i=1}^{M} P(A_i) + 2\sum_{i<j} P(A_i \cap A_j)$$

$$= \frac{(M-1)^N}{M^{N-1}} + 2\binom{M}{2}\left(\frac{M-2}{M}\right)^N$$

$$\mathrm{Var}[X_1 + \cdots + X_M] = \frac{(M-1)^N}{M^{N-1}}\left(1 - \frac{(M-1)^N}{M^{N-1}}\right) + 2\binom{M}{2}\left(\frac{M-2}{M}\right)^N$$

6. Which of the following are distribution functions?

(a) $F(x) = \begin{cases} 1 - e^{-x^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$ (b) $F(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$ (c) $F(x) = e^x/(e^x + e^{-x})$

$$= \frac{1}{1 + e^{-2x}}$$

	→ 1 at ∞ → 0 at −∞	incr.	right-cont.
(a)	✓	✓	✓
(b)	✓	✓	✓
(c)	✓	✓	✓

(c): $F' = -(1 + e^{-2x})^{-2} e^{-2x}(-2) > 0$

(a): $F' = -e^{-x^2}(-2x) = 2xe^{-x^2} > 0$ for $x > 0$

7. Let X be a random variable and c is a real number such that $P(X = c) > 0$.
   (a) Show that the distribution function F_X is discontinuous at the point $x = c$.
   (b) Does the converse hold?

7(a) Suppose $P(X = c) = \delta > 0$

For $x < c$, $F(c) - F(x) = P(x < X \leq c) \geq P(X = c) > \delta$

so F is not (left-) cont.

(b) cl. $P(X < c) = \lim_{x \to c^-} F(x) = \lim_{n \to \infty} F(c - \frac{1}{n})$ by cont. of measure:

or for any seq. $\searrow 0$

follows since $\bigcup_n \{X \leq c - \frac{1}{n}\} = \{X < c\}$.

So if F_X is discont. at c, by monotonicity sided limits exist,

& by right-cont., we must have

$$P(X < c) = \lim_{x \to c^-} F_X(x) < \lim_{x \to c^+} F_X(x) = F_X(c) = P(X \leq c)$$

$$\Rightarrow \quad P(X = c) = P(X \leq c) - P(X < c) > 0.$$

8. Let X be a random variable with distribution function F_X and let $Y = \max\{0, X\}$. Show that Y is a random variable and find its distribution function F_Y in terms of F_X.

$P(Y \leq x) = P(0 \leq Y \leq x) = P(Y = 0) + P(0 < X \leq x) = F(0) + (P(X \leq x) - P(X \leq 0)) = F(x)$

for $x \geq 0$, otherwise 0.

$$Y^{-1}((a,b)) = \begin{cases} X^{-1}((a,b)) & 0 < a < b \\ X^{-1}([0,b)) & a < 0 < b \\ \emptyset & a < b < 0 \end{cases} \quad \text{so } Y \text{ is measurable.}$$

9. Let X be a random variable. A real number m is called a median of X if

$$P(X < m) \leq \frac{1}{2} \leq P(X \leq m).$$

(a) Show that every random variable has a median.

(b) Give an example of a random variable that has more than one median.

9a. Set $a = \inf\{x : F(x) \geq \frac{1}{2}\}$.

WTS $F(a^-) \leq \frac{1}{2} \leq F(a)$

($\leq$) $\exists a_n \mid F(a_n) \geq \frac{1}{2}$, $a_n \searrow a$.

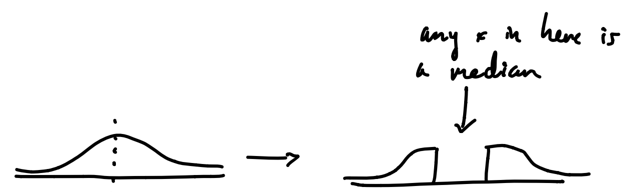
by R-cont., $F(a_n) \to F(a)$ so $F(a) \geq \frac{1}{2}$

($\leq$) If $F(a^-) > \frac{1}{2}$, $\exists \delta > 0 : x \in (a - \delta, a) \Rightarrow F(x) > \frac{1}{2}$

But then $(a - \delta, a) \subseteq \{x : F(x) \geq \frac{1}{2}\}$

so that $a \leq a - \delta$ ↯

9b. Can obtain an example from any symmetric dist.:



disc. 8

Review of some HW problems

1. Let X be a RV $: \to \mathbb{Z}$ s.t. $P(X=k) = p_k$ for $k \in \mathbb{Z}$. Show that F_X satisfies

$$F_X(b) - F_X(a) = p_{a+1} + p_{a+2} + \cdots + p_b$$

for $a, b \in \mathbb{Z}$ s.t. $a < b$. (What if a, b are in $\mathbb{R}$ instead?)

$$F_X(b) - F_X(a) = P(X \le b) - P(X \le a) = P(a < X \le b) = p_{a+1} + p_{a+2} + \cdots + p_b.$$

for $a, b \in \mathbb{R}$, $a < b$, we have

$$F_X(b) - F_X(a) = P(a < X \le b) = p_{a^*} + p_{a^*+1} + \cdots + p_{\lfloor b \rfloor} \quad \text{where } a^* = \begin{cases} a+1 & \text{if } a \in \mathbb{Z} \\ \lceil a \rceil & \text{if } a \notin \mathbb{Z} \end{cases}$$

3. $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy \overset{\text{polar}}{=} \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\, r\,d\theta\,dr$$

$$= 2\pi \int_0^{\infty} r e^{-r^2}\,dr = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} = \pi \quad \Rightarrow \quad I = \sqrt{\pi}.$$

To get Gaussian normalization, do a change of vars.

4. Suppose X is a cont. RV w/ density

$$f_X = \begin{cases} 0 & \text{if } x < -1 \\ c(1-x^2) & \text{if } -1 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}$$

(a) Find c.
(b) Find F_X
(c) $P(X > 0) = ?$
$P(X > 1) = ?$

(a) $1 = c\int_{-1}^{1}(1-x^2)\,dx = \frac{4}{3}c \Rightarrow c = \frac{3}{4}$

(b) $$F_X = \begin{cases} 0 & (x < -1) \\ \frac{3}{4}\int_{-1}^{x}(1-y^2)\,dy & (-1 \le x \le 1) \leftarrow \frac{3}{4}\left((x+1) - \frac{x^3+1}{3}\right) \\ 1 & (x > 1) \end{cases}$$

(c) $P(X > 0) = 1 - F_X(0) = 1 - \frac{3}{4}\left(1 - \frac{1}{3}\right) = 1 - \frac{3}{4}\frac{2}{3} = \frac{1}{2}$ (can also see by inspection)

$P(X > 1) = 0$

5(B) $\quad X \sim \mathrm{Exp}(\lambda) \quad \left(\Rightarrow f_X(t) = \lambda e^{-\lambda t},\ F_X(t) = 1 - e^{-\lambda t} \text{ for } t > 0\right)$

$B = e^X$

for $t > 0$, $P(e^X \le t) = P(X \le \log t) = F_X(\log t) = 1 - e^{-\lambda \log t} = 1 - t^{-\lambda}$

6. Suppose $X \sim \mathrm{Geom}(p)$, $Y \sim \mathrm{Geom}(r)$ be indep. RVs. Show that $Z := \min(X, Y) \sim \mathrm{Geom}(p + r - pr)$.

$$\sum_{j \ge z} (1-p)^{j-1} p$$

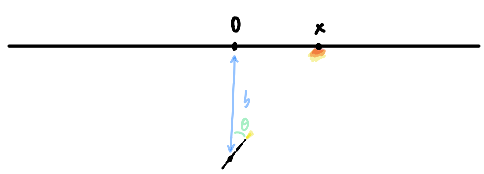
We have $\min(X, Y) > z$ iff $X > z,\ Y > z$

so
$$\begin{aligned}
P(Z > z) &= P(X > z)\,P(Y > z) \quad \text{by independence} \\
&= \underbrace{\left(\sum_{i=z+1}^{\infty} (1-p)^{i-1} p\right)}_{(1-p)^z p \sum_{i=0}^{\infty} (1-p)^i \,=\, (1-p)^z} \left(\sum_{j=z+1}^{\infty} (1-r)^{j-1} r\right) \\
&= (1-p)^z (1-r)^z \\
&= ((1-p)(1-r))^z \\
&= (1 - (p + r - rp))^z
\end{aligned}$$

Prob. (Bulmer) A machine gun is placed at dist. $b > 0$ from an infinite straight wall. The MG is pointed with angle off $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, uniformly randomly. Show that X, the dist. from the center of the wall to the impact point, has a Cauchy dist.

$$f(x) = \frac{b}{\pi (b^2 + x^2)}.$$

Soln.



Observe that $\tan \theta = \frac{x}{b}$, so that $P(X \le x) = P\left(-\frac{\pi}{2} \le \theta \le \arctan \frac{x}{b}\right)$

$$= \frac{\arctan \frac{x}{b} + \frac{\pi}{2}}{\pi}.$$

Hence

$$f(x) = \frac{d}{dx} F(X \le x) = \frac{1}{\pi} \cdot \frac{1}{1 + \left(\frac{x}{b}\right)^2} \cdot \frac{1}{b} = \frac{1}{\pi} \cdot \frac{b}{b^2 + x^2}$$

N.b. this dist. has infinite mean

disc. 9 **Note:** #3 requires some assumption, e.g. at most one of the integrals is infinite

4d $Y \sim \text{Poisson}(\lambda)$

$$P[Y \geq \alpha] = P[X \leq \lambda] \qquad X \sim \text{Gamma}(\alpha, 1)$$

$$P[Y \geq \alpha] = \sum_{k \geq \alpha} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda}\left(e^{\lambda} - \sum_{k=0}^{\alpha-1} \frac{\lambda^k}{k!}\right) = 1 - e^{-\lambda}\sum_{k=0}^{\alpha-1}\frac{\lambda^k}{k!}$$

$$\frac{1}{\Gamma(\alpha)}\int_0^{\lambda} x^{\alpha-1}e^{-x}\,dx \overset{y = x/\lambda}{=} \frac{1}{\Gamma(\alpha)}\int_0^1 (\lambda y)^{\alpha-1} e^{-\lambda y}\,dy \cdot \lambda$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)}\int_0^1 y^{\alpha-1}e^{-\lambda y}\,dy$$

$$P[X_{\alpha+1} \leq \lambda] = \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1 y^{\alpha}e^{-\lambda y}\,dy = \frac{\lambda^{\alpha}\alpha}{\Gamma(\alpha+1)}\left(\int_0^1 y^{\alpha-1}e^{-\lambda y}\,dy - e^{-\lambda}\right)$$

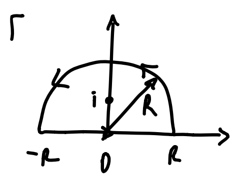
$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)}\int_0^1 y^{\alpha-1}e^{-\lambda y}\,dy - \frac{\lambda^{\alpha}}{\Gamma(\alpha)}e^{-\lambda}$$

$$= P[X_{\alpha} \leq \lambda] - P[Y = \alpha]$$

$$P[Y \geq \alpha + 1] = P[Y \geq \alpha] - P[Y = \alpha]$$
$$= P[X_{\alpha} \leq \lambda] - P[Y = \alpha] = P[X_{\alpha+1} \leq \lambda].$$

1d

Γ



$$\int_{\Gamma} \frac{e^{2iz}}{1+z^2}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{e^{2iz}}{1+z^2} = 2\pi i \lim_{z\to i}\frac{(z-i)e^{2iz}}{(z-i)(z+i)} = 2\pi i\,\frac{e^{-2}}{2i} = \frac{\pi}{e^2}$$

$$\Rightarrow\; = \int_{-R}^{R}\frac{e^{2ix}}{1+x^2}\,dx + \int_{\substack{|z|=R \\ \operatorname{Im} z > 0}} \underbrace{\frac{e^{2iz}}{1+z^2}}\,dz$$

$\searrow$ $O\left(\frac{1}{R^2}\right)$ (need $0 \leq \theta \leq \pi$)

so this integral is $O\left(\frac{1}{R}\right)$

$$2\pi i \Big/ \frac{e^{-2i(-i)}}{-\cancel{2i}}$$

$$= -\pi e^{-2}$$

So as $R \to \infty$, we get

$$\int_{-\infty}^{\infty}\frac{e^{2ix}}{1+x^2}\,dx = \frac{\pi}{e^2} \quad \& \text{ sim. } \int_{-\infty}^{\infty}\frac{e^{-2ix}}{1+x^2}\,dx = \frac{\pi}{e^2}$$

(use $-i$ and check orientation!)

so $$\int_{-\infty}^{\infty}\frac{\sin^2 x}{1+x^2}\,dx = -\frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{2ix}+e^{-2ix}-2}{1+x^2}\,dx$$

$$= -\frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{2ix}}{1+x^2}\,dx - \frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{-2ix}}{1+x^2}\,dx + \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$$

$$= -\frac{1}{4}\frac{\pi}{e^2} - \frac{1}{4}(+\pi e^{-2}) + \frac{1}{2}\pi$$

$$= \frac{1}{4}\pi(-2e^{-2}) + \frac{1}{2}\pi = \frac{\pi}{2}\left(1 - \frac{1}{e^2}\right)$$

HW9

2(i) $f(x,y) = \begin{cases} c(x^2 + \frac{1}{2}xy) & 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{oth.} \end{cases}$

(a) $1 = c\int_0^2\int_0^1 (x^2 + \frac{1}{2}xy)\, dx\, dy = c\left(\frac{2}{3} + \frac{1}{2}\right) = \frac{7}{6}c$

(b) $P(X \le x, Y \le y) = \frac{6}{7}\int_0^x\int_0^y s^2 + \frac{1}{2}st\ ds\, dt = \frac{6}{7}\left(\frac{x^2y^2}{8} + \frac{x^3y}{3}\right) \qquad 0 < x < 1,\ 0 < y < 2$

(c) $F_Y(y) = \frac{6}{7}\int_0^1 \left(x^2 + \frac{1}{2}xy\right) dx = \frac{6}{7}\left(\frac{1}{3} + \frac{1}{4}y\right)$

$F_X(x) = \frac{6}{7}\int_0^2 \left(x^2 + \frac{1}{2}xy\right) dy = \frac{6}{7}\left(x + 2x^2\right)$

(d) No e.g. check $x = y = 1$

6. X, Y joint density $f(x,y) = \begin{cases} \frac{1}{2}(x+y)e^{-x-y} & x, y \ge 0 \\ 0 & \text{oth.} \end{cases}$

Density of $X + Y$?

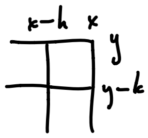
$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z - x)\, dx$$

as $P(X+Y \le z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f(x,y)\, dy\, dx \underset{\substack{u = x \\ v = x+y}}{=} \int_{-\infty}^{z}\int_{-\infty}^{\infty} f(u, v-u)\, du\, dv$

then differentiate.

$$f_{X+Y}(z) = \int_0^z \frac{1}{2}(x + z - x)e^{-x-(z-x)}\, dx = \frac{1}{2}z^2e^{-z}$$

Re: 7



$F(x,y) - F(x, y-k) - F(x-h, y) + F(x-h, y-k) \ge 0$

$F(x,y) + F(x-h, y-k) \ge F(x, y-k) + F(x-h, y)$