132 disc. 1

Bertrand Stone
bertrand.stone@math.ucla.edu
MS 2905
OH: TBD
bertrandstone.com/ta

Complex numbers $\mathbb{C}$: numbers of the form $z = a + bi$ where $a, b \in \mathbb{R}$

(a, b)
real part ($\operatorname{Re} z$) — imaginary part ($\operatorname{Im} z$)

difference between $\mathbb{C}$ and $\mathbb{R}^2$:

$\mathbb{C}$ has a product:

$$(a+bi)(c+di) = ac + ibc + ida + bdi^2 = (ac-bd) + (ad+bc)i$$

$$(a,b)(c,d) = (ac-bd, ad+bc)$$

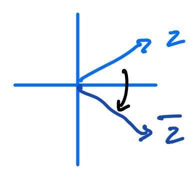
and you can do division. Define the norm or modulus of $z = a+bi$ by $|z| = \sqrt{a^2+b^2}$.

What's the same? distance:

$$z = a+bi, \quad w = c+di$$

$$|z-w| = \sqrt{(\operatorname{Re} z - \operatorname{Re} w)^2 + (\operatorname{Im} z - \operatorname{Im} w)^2} = \sqrt{(a-c)^2 + (b-d)^2} = \|(a,b) - (c,d)\|$$

The conjugate of $z = a+bi$ is $\bar{z} = a - bi$. We have:



$$\overline{z+w} = \bar{z} + \bar{w}, \quad \overline{zw} = \bar{z}\bar{w}, \quad \operatorname{Re} z = \frac{z+\bar{z}}{2}, \quad \operatorname{Im} z = \frac{z-\bar{z}}{2i}$$

p. 5 # 14 $\quad \operatorname{Re}(iz) = \dfrac{iz + \overline{iz}}{2} = \dfrac{iz - i\bar{z}}{2} = \dfrac{z-\bar{z}}{2} i = -\dfrac{z-\bar{z}}{2i} = -\operatorname{Im} z$

p. 6 # 22 $\quad$ Roots of $z^4 - 16 = 0$:

factor as $(z^2)^2 - 4^2 = (z^2-4)(z^2+4) = (z-2)(z+2)(z^2+4)$

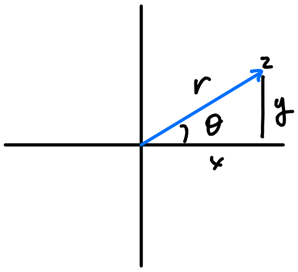
$$\Rightarrow z = \pm 2, \pm 2i$$

Fundamental theorem of algebra The polynomial $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ ($a_n \neq 0$, $n \geq 1$) has exactly n roots in $\mathbb{C}$.

[Hard part is showing that a root exists: we'll be able to do this later in the course, e.g. using Liouville's thm.]

Note: we count with multiplicity, so e.g. $z^2 = 0$ has two solutions.

Polar form.



$$z = x + iy$$

$$r = |z| = \sqrt{x^2 + y^2}$$

if $x, y > 0$,

$$\theta = \arctan \frac{y}{x} + 2\pi k \quad (k \in \mathbb{Z}).$$

$\theta =$ "$\arg z$"

So the polar form is not unique. We write $z = r(\cos\theta + i \sin\theta)$.

Write $\theta = \underline{\operatorname{Arg} z}$ to choose the θ s.t. $-\pi < \theta \le \pi$.

Complex exponential

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Key property:
$$e^{z+w} = e^z e^w$$

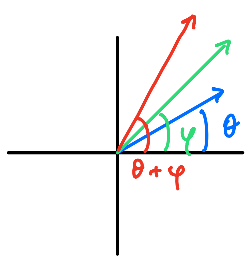
Def. $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}.$

Then:
$$e^{i\theta} = \cos\theta + i \sin\theta \qquad (\theta \in \mathbb{R})$$

so we can write
$$z = re^{i\theta}$$

<u>Multiplication</u>: $z = re^{i\theta}, \; w = se^{i\varphi}$:

$$zw = rse^{i\theta}e^{i\varphi} = rse^{i(\theta+\varphi)}$$



so when we multiply, we add the angles and multiply the lengths.

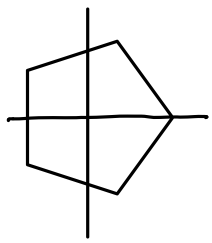
Roots of unity

The eqn. $z^n = 1$ has n solutions, given by

$$z = e^{2\pi k i/n} \qquad (k = 0, 1, 2, \ldots, n-1)$$

$$= \cos\frac{2\pi k}{n} + i \sin\frac{2\pi k}{n}$$

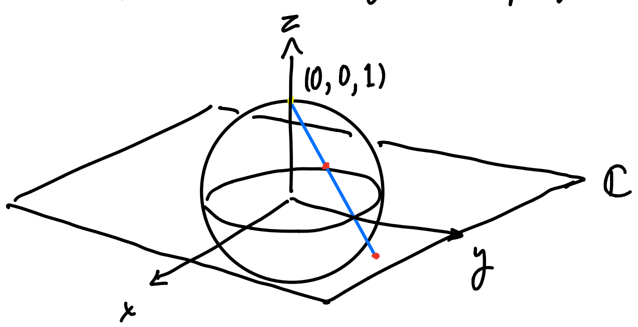
The roots of unity form the vertices of a regular n-gon inscribed in the unit circle.



$(n = 5)$

132 disc. 2

Riemann sphere / stereographic projection



Let $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$.

For $(x_1, x_2, x_3) \in S^2$ exc. $(0,0,1)$, assoc. the complex # the blue line hits when going through $(0,0,1)$, (x_1, x_2, x_3). That is,

$$(0,0,1) + t(x_1, x_2, x_3 - 1) = (x, y, 0)$$

$$\Rightarrow 1 + t(x_3 - 1) = 0$$

$$\Rightarrow t(x_3 - 1) = -1$$

$$\Rightarrow t = \frac{1}{1 - x_3}$$

$$\Rightarrow (x, y) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right)$$

i.e., $\boxed{z = \frac{x_1 + i x_2}{1 - x_3}}$

To go the other way, note that

$$|z|^2 = \left(\frac{|x_1 + i x_2|}{1 - x_3} \right)^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 - x_3^2}{(1 - x_3)^2}$$

$$= \frac{(1 - x_3)(1 + x_3)}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3}$$

$$\Rightarrow (1 - x_3)|z|^2 = 1 + x_3$$

$$\Rightarrow |z|^2 - x_3 |z|^2 = 1 + x_3 \Rightarrow |z|^2 - 1 = x_3(1 + |z|^2)$$

$$\Rightarrow x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

so $\dfrac{z+\bar{z}}{2} = \operatorname{Re} z = \dfrac{x_1}{1-x_3}$

$$\Rightarrow \quad x_1 = \frac{z+\bar{z}}{2}(1-x_3) = \frac{z+\bar{z}}{2}\cdot\frac{2}{|z|^2+1}$$

$$\boxed{x_1 = \frac{z+\bar{z}}{|z|^2+1}}$$

& sim.

$$x_2 = \frac{z-\bar{z}}{i(|z|^2+1)}$$

for completeness

$$x_3 = \frac{|z|^2-1}{|z|^2+1}$$

Now, $(0,0,1)$ corresponds to the "point at infinity": we will write ∞

Thus, the whole S^2 corresponds to $\mathbb{C} \cup \{\infty\}$.

We now show:

<u>Prop.</u> Stereographic projection takes circles in $\mathbb{C}$ to circles on S^2.

<u>Important note:</u> Lines in $\mathbb{C}$ are circles through ∞.

<u>Pf.</u> Any circle on the sphere lies in some plane

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0$$

where we can assume $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$, $0 \le \alpha_0 < 1$.

In terms of z,

$$\alpha_1(z+\bar{z}) - i\alpha_2(z-\bar{z}) + \alpha_3(|z|^2-1) = \alpha_0(|z|^2+1)$$

or, with $z = x+iy$,

$$2\alpha_1 x - i\alpha_2(2iy) + \alpha_3(x^2+y^2-1) = \alpha_0(x^2+y^2+1)$$

$$\Rightarrow \quad (\alpha_0-\alpha_3)(x^2+y^2) - 2\alpha_1 x - 2\alpha_2 y + \alpha_0 + \alpha_3 = 0$$

so if $\alpha_0 \neq \alpha_3$, this is a circle, and if $\alpha_0 = \alpha_3$, this is a line.

Chordal distance :

If $z, z' \in \mathbb{C}$, what's the distance $d(z, z')$ between their projections (x_1, x_2, x_3), (x_1', x_2', x_3') on the sphere?

We have
$$(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2$$
$$= 2 - 2(x_1x_1' + x_2x_2' + x_3x_3')$$

where
$$x_1x_1' + x_2x_2' + x_3x_3' = \frac{z+\bar z}{|z|^2+1}\cdot\frac{z'+\bar z'}{|z'|^2+1} + \frac{z-\bar z}{i(|z|^2+1)}\cdot\frac{z'-\bar z'}{i(|z'|^2+1)} + \frac{|z|^2-1}{|z|^2+1}\cdot\frac{|z'|^2-1}{|z'|^2+1}$$
$$= \frac{(z+\bar z)(z'+\bar z') - (z-\bar z)(z'-\bar z') + (|z|^2-1)(|z'|^2-1)}{(|z|^2+1)(|z'|^2+1)}$$
$$= \frac{1}{\cdots}\Big(\cancel{zz'} + 2\bar z z' + 2\bar z' z + \cancel{\bar z\bar z'} - \cancel{zz'} + \cancel{\bar z z'} + \cancel{\bar z' z} - \cancel{\bar z\bar z'}$$
$$+ |z|^2|z'|^2 + 1 + |z'|^2 + |z|^2\Big)$$
$$-2|z'|^2 - 2|z|^2$$
$$= \frac{(1+|z|^2)(1+|z'|^2) - 2|z-z'|^2}{(|z|^2+1)(|z'|^2+1)}$$

$\Rightarrow$

$$d(z, z') = \frac{2|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}}$$ the <u>chordal distance</u>.

132 disc. 3 10/10/19

We say $f:\mathbb{C}\to\mathbb{C}$ is (complex) differentiable at z_0 if

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists.

Ex. $f(z) = \bar{z}$ is nowhere differentiable:

$$\frac{\overline{z+h} - \bar{z}}{h} = \frac{\bar{h}}{h}$$

as $h\to 0$ along $\mathbb{R}$, get 1

as $h \to 0$ along im. axis, get -1:

$$\frac{\overline{iy}}{iy} = \frac{-iy}{iy} = -1$$

Key idea: allowing $h\to 0$ from any direction makes complex differentiability a very strong condition.

Note: usual rules of calculus hold (sum, product, quotient, chain rule)

We say f is analytic or holomorphic in an open set Ω if f is differentiable for each $z\in\Omega$.

But, we can also think in terms of real & imaginary parts:

$$z = x+iy, \qquad f(z) = u(x,y) + iv(x,y)$$

Recall from calc:

We say $\vec{F}:\mathbb{R}^2 \to$ is differentiable in the real sense if

$$\lim_{\vec{h}\to\vec{0}} \frac{\|\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x}) - J\vec{h}\|}{\|\vec{h}\|} = 0 \quad \text{for some } 2\times 2 \text{ matrix } J$$

which we call the Jacobian matrix.

In this case, the partials exist &

$$J = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix}.$$

So, what's the difference?

Suppose $f = u + iv$ is complex differentiable at $z_0 = (x_0, y_0)$.

We know
$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$
$$= \lim_{\substack{h_1 \to 0 \\ h_1 \in \mathbb{R}}} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} = \frac{\partial f}{\partial x}(z_0)$$

& sim. using $h = ih_2$, we get
$$f'(z_0) = \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2} = \frac{1}{i}\frac{\partial f}{\partial y}(z_0)$$

So, $\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$ at (x_0, y_0).

Now since $f = u + iv$,
$$\frac{\partial(u + iv)}{\partial x} = -i\frac{\partial(u+iv)}{\partial y}$$
$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$
$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \quad \text{at } z_0$$

These are the Cauchy-Riemann equations.

<u>Ex.</u> $f(x, y) = \sqrt{|x||y|}$. This f satisfies the C-R eqns. at $(0,0)$ but is not differentiable there.

$u(x, y) = \sqrt{|x||y|}$
$$\frac{\partial u}{\partial x}(0,0) = \lim_{\substack{h \to 0 \\ \text{real}}} \frac{\sqrt{|0+h||0|} - \sqrt{|0||0|}}{h} = 0$$
$\frac{\partial u}{\partial y}(0,0) = 0$ similarly

$v = 0 \Rightarrow \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} = 0$

We also have
$$\lim_{h_1 + ih_2 \to 0} \frac{\sqrt{|0+h_1||0+h_2|} - \sqrt{|0||0|}}{h_1 + ih_2} = \lim_{h_1 + ih_2 \to 0} \frac{\sqrt{|h_1||h_2|}}{h_1 + ih_2}$$

so if $h_1 = h_2 \to 0$, $\lim_{h_1 \to 0} \frac{\sqrt{|h_1|^2}}{(1+i)h_1} = \frac{1}{1+i} \lim_{h_1 \to 0} \frac{|h_1|}{h_1}$ which does not exist.

We also have

<u>Thm.</u> Suppose $f = u + iv$ is defined on an open set Ω.

If u, v are continuously differentiable & satisfy the C-R eqns. in Ω, then f is analytic on Ω.

Pf. We have

$$u(x+h_1, y+h_2) - u(x,y) = \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + |h| \varphi_1(h)$$

$$v(x+h_1, y+h_2) - v(x,y) = \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + |h| \varphi_2(h)$$

(MVT/continuity)

wh. $h = h_1 + ih_2$, $\varphi_j \to 0$ as $|h| \to 0$.

Then C-R $\Rightarrow$

$$f(z+h) - f(z) = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) h + |h|(\varphi_1 + i\varphi_2)(h)$$

so f is hol. in Ω $\lrcorner$

If $J = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

$\det J \neq 0$

$$= \underbrace{\sqrt{a^2+b^2}}_{C} \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix}$$

$$= C \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{for some } \theta$$

By C-R, $\det J = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

$$|f'(z_0)|^2 = \left|\frac{\partial f}{\partial x}\right|^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

so $f'(z_0) \neq 0 \Rightarrow$ locally f is a rotation & scaling (no reflections).

That is, f is <u>orientation-preserving</u>.

(This is "why" $f(z) = \bar{z}$ is not analytic).

132 disc. 4 10/17/19

logs

Want an inverse to $z = e^w$

$\log z = w : e^w = z$

In $\mathbb{R}$, this is fine.

Problem: $e^{2\pi i} = 1 \Rightarrow e^{(y+2\pi)i} = e^{yi}$. If $z = re^{i\theta}$,

$$\log z = \log |z| + i \arg z$$

satisfies

$$e^{\log z} = e^{\log |z|} e^{i \arg z} = e^{\log r} e^{i(\theta + \overset{\text{(any } k)}{2\pi k})} = re^{i\theta} = z.$$

But this log has infinitely many possible values.

We thus define the principal branch

$$\operatorname{Log} z = \log |z| + i \operatorname{Arg} z \quad \text{where we recall, } -\pi < \operatorname{Arg} z \le \pi$$

Because the Arg jumps by 2π as we move over $\mathbb{R}^-$, we will define Log on a domain with a <u>branch cut</u> : specifically, we remove $\mathbb{R}_{\le 0} = \{y \in \mathbb{R} : y \le 0\}$.

<u>Prop.</u> Log is analytic in $\{z \in \mathbb{C} : z \in \mathbb{R}_{\le 0}\}$, with derivative $1/z$.

<u>Pf.</u> We try direct computation:

$$\lim_{z \to z_0} \frac{\operatorname{Log} z - \operatorname{Log} z_0}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{e^w - e^{w_0}} = \lim_{w \to w_0} \frac{1}{\frac{e^w - e^{w_0}}{w - w_0}} = \frac{1}{e^{w_0}} = \frac{1}{z_0}$$

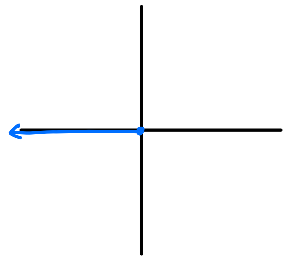
$z = e^w$, $z_0 = e^{w_0}$

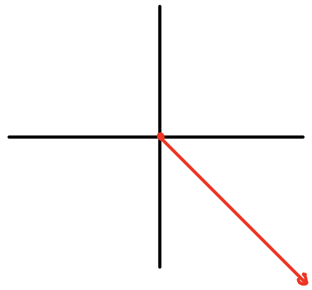
This works as long as

here, need the branch cut for continuity of log → (i) $z \to z_0 \Leftrightarrow w \to w_0$: this is just by continuity of exp and Log

(ii) $z \neq z_0 \Rightarrow w \neq w_0$: this is immediate.

We can also use other branch cuts:





In the second case, we use an arg w/

$$-\frac{\pi}{4} < \arg z < 2\pi - \frac{\pi}{4} = \frac{7}{4}\pi.$$

So, with this $\log = \log_{-\pi/4}$

$$\log(-1) = \log|1| + i\arg(-1)$$
$$= 0 + i\pi = i\pi$$

which indeed satisfies $e^{i\pi} = -1$.

In fact, this is exactly why we might need other branch cuts.

<u>Ex. 2 (from book)</u>

Find a branch of $\log(z^3 - 2)$ that's analytic at $z = 0$.

Want a log that's analytic at -2. E.g. we can use $\log_{-\pi/4}$.

In fact, we can go further:

<u>Thm.</u> If Ω is open, simply connected, & $0 \notin \Omega$, then there is an analytic branch of log on Ω.

<u>Thm.</u>
Same, for any entire function that's everywhere nonzero, like e^z

<u>Problem</u> List all entire functions f s.t.

$$f^2 + (f')^2 = 1.$$

Prove that your list is exhaustive.

(We'll do the proof next time)

132 disc. 5 10/24/19

Start w/ answer to problem from last time.

Complete list is : $f = \pm 1$, $\cos(z-c)$ for $c \in \mathbb{C}$.

Pf. If f is constant, then $f' = 0$, so $f^2 + (f')^2 = 1 \Rightarrow f = \pm 1$.

Suppose f is not constant. Write $g = f'$. We have

$$\underbrace{f^2 + g^2}_{= (f-ig)(f+ig)} = 1 \qquad (*)$$

So $f-ig$, $f+ig$ have no zeros. Thus $\exists h$ hol. in $\mathbb{C}$ s.t.

$f - ig = e^h$. Then $(*) \Rightarrow f + ig = e^{-h}$.

Adding the last two eqns.,

$$f = \frac{e^h + e^{-h}}{2}$$

& similarly $-2ig = f - ig - (f+ig) = e^h - e^{-h}$

$$\Rightarrow \quad g = -\frac{e^h - e^{-h}}{2i}.$$

But we also know $g = f' = \dfrac{d}{dz} \dfrac{e^h + e^{-h}}{2}$

$$= \frac{e^h h' + e^{-h}(-h')}{2}$$

$$= h'\left(\frac{e^h - e^{-h}}{2}\right)$$

hence $\quad -\dfrac{e^h - e^{-h}}{2i} = h'\left(\dfrac{e^h - e^{-h}}{2}\right).$

Want to cancel $\dfrac{e^h - e^{-h}}{2}$: can do this since $e^h \not\equiv e^{-h}$ except at isolated points : otherwise we would have $e^h \equiv e^{-h}$ by analytic continuation, hence $f \equiv$ const., a contradiction.

Thus, $-\frac{1}{i} = h' \Rightarrow h'(z) = i \Rightarrow h(z) = iz + C$

$= iz - ic$ for $c = \dfrac{C}{-i}$

That is, $f = \dfrac{e^h + e^{-h}}{2} = \dfrac{e^{i(z-c)} + e^{-i(z-c)}}{2} = \cos(z-c).$

Midterm 1

1. $(z-1)^{2019} = z^{2019} \Rightarrow \left(\frac{z-1}{z}\right)^{2019} = 1$

$$\Rightarrow \frac{z-1}{z} = e^{2\pi i \frac{k}{2019}} \qquad 0 \le k < 2019$$

$$\Rightarrow z - 1 = z e^{2\pi i k/2019}$$

$$\Rightarrow z\left(1 - e^{2\pi i k/2019}\right) = 1$$

$$\Rightarrow z = \frac{1}{1 - e^{2\pi i k/2019}} \qquad \text{for } 1 \le k < 2019$$

2. Let ∂D_2 be the boundary of $\{|z|<2\}$. Then

(i)
$$\left|\frac{1}{z^2-1}\right| \le \frac{1}{3}$$

(ii) does this hold in $|z| < 2$?

(i) eqv. to $|z^2 - 1| \ge 3$. We have

$$|z^2 - 1| \ge |z^2| - 1 = 4 - 1 = 3 \quad \text{by rev. } \Delta\text{-ineq.}$$

(ii) No; let $z = 0$.

3. Suppose f is entire w/ $f = u + iv$ and $u(x,y)v(x,y) = 2019$.

real-valued

Show that f is constant.

We have $\partial_x(uv) = 0$ and $\partial_y(uv) = 0$ and $\begin{cases} u_x = v_y \\ -u_y = v_x \end{cases}$ by Cauchy-Riemann

$$\underset{\displaystyle u_x v + u v_x}{\partial_x(uv)} \qquad \underset{\displaystyle u_y v + u v_y}{\partial_y(uv)}$$

so
$$\underset{\displaystyle v_y v - u u_y}{u_x v + u v_x} = 0 = \underset{\displaystyle -v_x v + u u_x}{u_y v + u v_y} \qquad u u_x = v v_x$$

$$\Rightarrow u u_x = v v_x, \quad u u_y = v v_y.$$

Thus, $vv_y = uu_y$

$$\left(\tfrac{1}{2}v^2\right)_y = \left(\tfrac{1}{2}u^2\right)_y \Rightarrow u^2 = v^2 + C(x)$$
$$\Rightarrow u^2 - v^2 = C(x)$$

Similarly, $u^2 - v^2 = C(y)$ using the other eqn.

Thus, $u^2 - v^2 = C$.

We thus have
$$(u+iv)^2 = u^2 + 2iuv - v^2$$
$$= (u^2 - v^2) + 2iuv$$
$$= C + 2i \cdot 2019$$

so f^2 is const.

hence f is constant.

<u>Prof. Salazar's soln.</u> We get $\begin{cases} u_x v + u v_x = 0 \\ u_y v + u v_y = 0 \end{cases}$ as above. Then use C-R (for 2nd) & multiply by v, u resp.:

$$\begin{cases} -v_x v^2 + u u_x v = 0 \\ u_x v u + u^2 v_x = 0 \end{cases} \Rightarrow v_x(u^2 + v^2) = 0.$$

Similarly $u_x(u^2+v^2) = 0$. Thus, since $uv = 2019 \Rightarrow u^2 + v^2 > 0$,

we see that $u_x \equiv 0$, $v_x \equiv 0$ so C-R again $\Rightarrow$ f is constant.

<u>4.</u> Note that

$$z \neq z_0, \quad \frac{f(z)}{g(z)} = \frac{f(z) - f(z_0)}{z - z_0} \frac{z - z_0}{g(z) - g(z_0)} \longrightarrow \frac{f'(z_0)}{g'(z_0)} \quad \text{as } z \to z_0.$$

Need $g \neq 0$ near z_0. [Oth. $g \equiv 0$ by continuation]

Prof. Salazar's reason: $g = 0$ near $z_0 \Rightarrow \dfrac{g(z) - g(z_0)}{z - z_0} = 0 \Rightarrow g'(z_0) = 0$ ↯

<u>5.</u> We have $\operatorname{Re} \frac{1}{z} = \operatorname{Re} \frac{1}{x+iy} = \operatorname{Re} \frac{x - iy}{x^2+y^2} = \frac{x}{x^2+y^2}$ & sim. $\operatorname{Im} \frac{1}{z} = \frac{y}{x^2+y^2}$.

Write $\frac{1}{z} = a + ib$. Then

$$e^a \cos b = e^a \frac{e^{ib} + e^{-ib}}{2} = \frac{e^{a+ib} + e^{a-ib}}{2},$$

but on the other hand,

$$\operatorname{Re} e^{1/z} = \frac{e^{1/z} + e^{1/\bar{z}}}{2} = \frac{e^{a+ib} + e^{a-ib}}{2}.$$

So $\varphi = \operatorname{Re}(e^{1/z})$ is harmonic.

132 disc. 6

Ex. The quintessential contour integral

$$\int_C \frac{1}{z-a}\, dz = 2\pi i$$

where C is a circ. $|z-a| = r$ w/ positive orientation.

Indeed,
$$\int_C \frac{dz}{z-a} = \int_a^b \frac{z'(t)\,dt}{z(t)-a} \qquad z(t) = a + re^{iT}, \quad 0 \le t \le 2\pi$$

$$= \int_0^{2\pi} \frac{i\cancel{re^{iT}}}{\cancel{re^{iT}}}\, dt = 2\pi i.$$

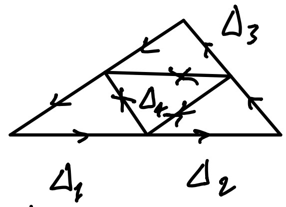
Now, follow Rudin for local Cauchy theorems. First do: $\int_\gamma f = 0$ if $f = F'$, γ a loop.

Cauchy's thm. for a triangle

Suppose f is analytic in an open set Ω, Δ is a clsd. triangle in Ω.
Then

$$\int_{\partial\Delta} f(z)\, dz = 0.$$

Pf. The proof is via bisection



Consider the subdivision above, so

$$\int_{\partial\Delta} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial\Delta_j} f(z)\, dz.$$

Hence

$$\left|\int_{\partial\Delta} f(z)\,dz\right| \leq 4\left|\int_{\partial\Delta j} f(z)\,dz\right| \quad \text{for some } 1 \leq j \leq 4:$$

if not, $\left|\int_{\partial\Delta} f(z)\,dz\right| > 4\left|\int_{\partial\Delta i} f(z)\,dz\right|$ for all j, so

$$4\left|\int_{\partial\Delta_j}\right| < \left|\int f\right| \leq \left|\int_{\partial\Delta_1}\right| + \left|\int_{\partial\Delta_2}\right| + \left|\int_{\partial\Delta_3}\right| + \left|\int_{\partial\Delta_4}\right| \leq 4\left|\int_{\partial\Delta_j}\right|$$

$\Delta^{(1)} = \Delta_j$ — j s.t. this is largest.

Now, iterate, subdividing the triangle / w/ j chosen above:

$$\left|\int_{\partial\Delta} f(z)\,dz\right| \leq 4^n\left|\int_{\partial\Delta_j^{(n)}} f(z)\,dz\right|$$

So, get a seq. $\Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \Delta^{(3)} \supseteq \cdots$

hence there is a <u>unique</u> point $z_0 \in \bigcap_{j=1}^{\infty} \Delta^{(j)}$

since $\Delta^{(j)}$ is cpt. & $\operatorname{diam} \Delta^{(j)} \to 0$ as $j \to \infty$.

Since f is analytic, for all $\varepsilon > 0$,

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon$$

for z near z_0, hence in $\Delta^{(j)}$ for large j.

That is,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon|z - z_0|$$

So

$$\int_{\partial\Delta^{(j)}} f(z)\,dz = \underbrace{\int_{\partial\Delta^{(j)}} f(z) - f(z_0) - f'(z_0)(z-z_0)\,dz}_{} + \underbrace{\int_{\partial\Delta^{(j)}} f(z_0) + f'(z_0)(z-z_0)\,dz}_{=0 \text{ since this fn. has a primitive}}$$

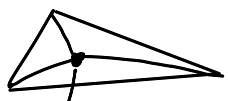
$$|\cdot| \leq \varepsilon \cdot \operatorname{perim} \Delta^{(j)} \cdot \operatorname{diam} \Delta^{(j)}$$
$$= \varepsilon \cdot 2^{-j} \operatorname{perim} \Delta \cdot 2^{-j} \operatorname{diam} \Delta$$

so that $\left|\int_{\partial\Delta} f\right| \leq 4^j \left|\int_{\partial\Delta^{(j)}} f\right| \leq 4^j \varepsilon\, 2^{-j} \operatorname{perim} \Delta\, 2^{-j} \operatorname{diam} \Delta$

$$= \varepsilon \cdot \operatorname{perim} \Delta \cdot \operatorname{diam} \Delta$$

so since $\varepsilon > 0$ was arbitrary, $\int_\Delta f = 0$.

Now, show this still holds if f is cont., f analytic in $\Omega \setminus \{p\}$:
consider case where p is a vertex, then reduce to this case:



p

Cauchy's thm. in a convex set

Suppose Ω is open, convex, f cont. in Ω, f analytic in $\Omega \setminus \{p\}$.
Then f has a primitive F in Ω, i.e. $f = F'$.
Hence

$$\int_\gamma f(z)\, dz = 0$$

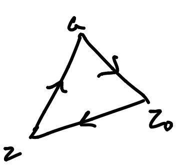
for every closed path γ in Ω.

Pf. Fix $a \in \Omega$. If $z \in \Omega$, the segment $[a, z] \subseteq \Omega$ since Ω is convex.
So define

$$F(z) = \int_{[a,z]} f(\zeta)\, d\zeta \qquad (z \in \Omega).$$

If $z, z_0 \in \Omega$, the triangle w/ vertices a, z, z_0 is in Ω.
So by Cauchy's thm. for a triangle,

$$F(z) - F(z_0) = \int_{[z_0, z]} f(\zeta)\, d\zeta \quad : \quad \text{indeed}$$



$$\int_{[z_0, z]} f + \int_{[z, a]} f + \int_{[a, z_0]} f = 0$$

so $$\int_{[z_0, z]} f = -\int_{[z, a]} f - \int_{[a, z_0]} f$$
$$= \int_{[a, z]} f - \int_{[a, z_0]} f = F(z) - F(z_0).$$

By continuity, for $\varepsilon > 0$, $\exists \delta > 0$: $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$

hence

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right|$$
$$= \left| \frac{1}{z - z_0} \int_{[z_0, z]} (f(\zeta) - f(z_0))\, d\zeta \right|$$
$$\leq \frac{1}{|z - z_0|} \sup |f(\zeta) - f(z_0)| \cdot \text{len}\, [z_0, z]$$
$$= \sup |f(\zeta) - f(z_0)| \leq \varepsilon \quad \text{if } \delta \text{ is as above.}$$

Thus $F' = f$.

132 disc. 7

Using Cauchy's theorem

Prop. For $\xi \in \mathbb{R}$,

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx$$

Pf. For $\xi = 0$, this is just

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx \qquad \text{(which we can do in polar coords.)}$$

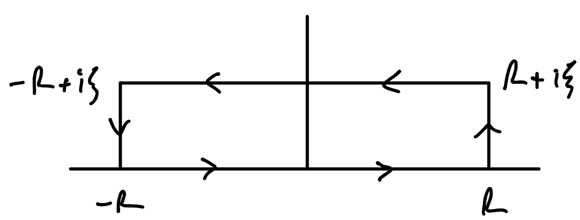
Now suppose $\xi > 0$.

Idea: the integrand suggests looking at

$$e^{-\pi(x+i\xi)^2} = e^{-\pi x^2} e^{-2\pi\xi i x} e^{\pi\xi^2}$$

so integrating RHS corresponds to integrating LHS, i.e. integrating $e^{-\pi z^2}$ over the segt. $[-R+i\xi, R+i\xi]$.

So, consider the contour Γ:



We have

$$\int_\Gamma e^{-\pi z^2}\, dz = 0 \qquad \text{by Cauchy's theorem}$$

On the other hand, it's

$$\underbrace{\int_{-R}^{R} e^{-\pi x^2}\, dx}_{\text{I}} + \underbrace{\int_{R}^{-R} e^{-\pi(x+i\xi)^2}\, dx}_{\text{II}} + \underbrace{\int_{[R,\, R+i\xi]} e^{-\pi z^2}\, dz}_{\text{III}} + \underbrace{\int_{[-R+i\xi,\, -R]} \text{''}}_{\text{IV}}$$

for the third integral,

$$\left|\int_{[R,\, R+i\xi]} e^{-\pi z^2}\, dz\right| = \left|\int_0^{\xi} \underbrace{e^{-\pi(R+iy)^2}}_{e^{-\pi(R^2+2iRy-y^2)}}\, i\, dy\right| \leq Ce^{-\pi R^2}, \quad \text{so}$$

III $\to 0$ as $R \to \infty$.

Sim. for IV.

We also know $\displaystyle\int_{-R}^{R} e^{-\pi x^2}\, dx \longrightarrow 1$ as $R \to \infty$

& $\displaystyle \text{II} = -\int_{-R}^{R} e^{-\pi(x+i\xi)^2}\, dx = -e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx$

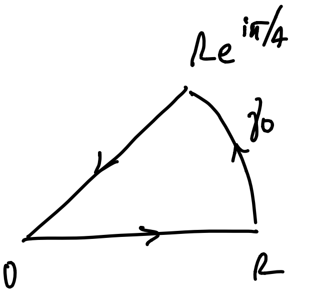
so as $R \to \infty$ we get

$$1 - e^{\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx + \underset{\text{III}}{0} + \underset{\text{IV}}{0} = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx = e^{-\pi\xi^2}.$$

Prop. $\displaystyle\lim_{x\to\infty}\int_0^x \sin(t^2)\, dt = \, ?$

Consider Γ:



and $f(z) = e^{-z^2}$

Then

$$\underset{\text{I}}{\int_0^R e^{-x^2}\, dx} + \underset{\text{II}}{\int_{\gamma_0} e^{-z^2}\, dz} + \underset{\text{III}}{\int_{[Re^{i\pi/4},\, 0]} e^{-z^2}\, dz} = \int_{\Gamma} e^{-z^2} = 0.$$

$\text{II} \to 0$ as $R \to \infty$.

$$\text{III} = \int_R^0 e^{-(e^{i\pi/4}y)^2} e^{i\pi/4}\, dy$$

$$= -e^{i\pi/4}\int_0^R e^{-iy^2}\, dy$$

$$\text{I} \to \int_0^\infty e^{-x^2}\, dx = \frac{1}{2}\int_{-\infty}^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

so as $R \to \infty$

$$\frac{\sqrt{\pi}}{2} - e^{i\pi/4}\int_0^\infty e^{-iy^2}\, dy = 0$$

$$\Rightarrow \int_0^\infty \underbrace{e^{-iy^2}}_{\cos(y^2) - i\sin(y^2)}\, dy = \frac{\sqrt{\pi}}{2} e^{-i\pi/4} = \frac{\sqrt{\pi}}{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \cos y^2\, dy = \mathrm{Re}\int_0^\infty e^{-iy^2}\, dy = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

$$\int_0^\infty \sin y^2\, dy = -\mathrm{Im}\int_0^\infty e^{-iy^2}\, dy = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

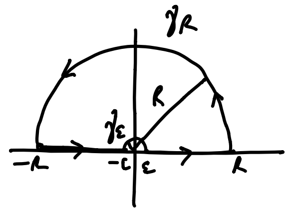
These are the Fresnel integrals.

132 disc. 8

Circular contour example

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} = ?$$

Consider $f(z) = \frac{e^{iz}}{z}$ $\qquad \Gamma$:



$$0 = \int_\Gamma f = \int_{\varepsilon<|x|<R} \frac{e^{ix}}{x}\, dx + \int_{\gamma_R} f \, dz + \int_{\gamma_\varepsilon} f \, dz$$

$$\left|\int_{\gamma_R} f\, dz\right| = \left|\int_0^\pi \frac{e^{i(R\cos\theta + iR\sin\theta)}}{\cancel{Re^{i\theta}}}\, i\cancel{Re^{i\theta}}\, d\theta\right| \le \int_0^\pi e^{-R\sin\theta}\, d\theta$$

$$0 \le e^{-R\sin\theta} \le 1 \text{ in } [0,\pi], \text{ so}$$

$$\lim_{R\to\infty} \int_0^\pi e^{-R\sin\theta}\, d\theta = \int_0^\pi \lim_{R\to\infty} e^{-R\sin\theta}\, d\theta = 0 \qquad (\text{by DCT}).$$

On the other hand,

$$\int_{\gamma_\varepsilon} f\, dz = -\int_0^\pi \frac{e^{i\varepsilon\cos\theta} e^{-\varepsilon\sin\theta}}{\cancel{\varepsilon e^{i\theta}}}\, i\cancel{\varepsilon e^{i\theta}}\, d\theta$$

$$= -i\int_0^\pi \left(e^{i\varepsilon\cos\theta} e^{-\varepsilon\sin\theta}\right) d\theta$$

The integrand is bdd. by $e^{-\varepsilon\sin\theta} \le 1$, so

$$\lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} f\, dz = -i\lim_{\varepsilon\to 0} \int_0^\pi e^{i\varepsilon\cos\theta} e^{-\varepsilon\sin\theta}\, d\theta$$

$$= -i\int_0^\pi \lim_{\varepsilon\to 0}\left(e^{i\varepsilon\cos\theta} e^{-\varepsilon\sin\theta}\right) d\theta = -i\pi \qquad (\text{by DCT}).$$

So sending $\varepsilon \to 0$, then $R\to\infty$, we get

$$0 = \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx + 0 - i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = i\pi$$

hence taking imaginary parts gives $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$

Prob. $\displaystyle\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = ?$ Hint: use Taylor series for one part of the integral

$$\mathrm{Re}\left(\frac{1-e^{ix}}{x^2}\right)$$

$f = \dfrac{1-e^{iz}}{z^2}$, Γ:

$$0 = \int_\Gamma f = \int_{\varepsilon<|x|<R} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_\varepsilon} f + \underbrace{\cancel{\int_{\gamma_R} f}}_{0 \text{ as } R\to\infty}$$

$$\int_{\gamma_\varepsilon} f = -\int_0^\pi \frac{1-e^{i\varepsilon e^{i\theta}}}{\varepsilon^2 e^{2i\theta}}\, i\varepsilon e^{i\theta}\,d\theta$$

$$\frac{1-e^{ix}}{x^2} = \frac{1-\left(1+ix+\frac{(ix)^2}{2}+\cdots\right)}{x^2} = -\frac{i}{x} + O(1)$$

$$\Rightarrow \int_{\gamma_\varepsilon} f = -\int_0^\pi \left(-\frac{i}{\varepsilon e^{i\theta}} + O(1)\right) i\varepsilon e^{i\theta}\,d\theta$$

$$= \underbrace{-\int_0^\pi \frac{1}{\cancel{\varepsilon e^{i\theta}}}\cancel{\varepsilon e^{i\theta}}\,d\theta}_{=-\pi} - \underbrace{\int_0^\pi O(1)\, i\varepsilon e^{i\theta}\,d\theta}_{|\cdot| \le O(1)\cdot\varepsilon \to 0 \text{ as } \varepsilon\to 0}$$

$\Rightarrow$ as $R\to 0$, $\varepsilon \to 0$,

$$0 = \int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}\,dx - \pi + 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}\,dx = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = \pi$$

Midterm 2

1. [5 pts] Suppose f and g are analytic in the simply connected domain D. If Γ is a simple closed contour completely contained in D, and if $f(z) = g(z)$ for all z along Γ, show that $f(z) = g(z)$ for every point inside of Γ.

   Hint: Let a be a point inside Γ. Apply Cauchy's integral formula to compute the value $h(a)$, where h is the difference function $h = f - g$.

We have

$$h(a) = \int_\Gamma \frac{\overbrace{f(\zeta) - g(\zeta)}^{=0 \text{ on } \Gamma}}{\zeta - a}\, d\zeta = 0.$$

2. [10 pts] Evaluate

$$\int_\Gamma \bar{z}\, dz,$$

where Γ is the arc of the parabola $y = x^2$ from $x = 0$ to $x = 1$.

$$z(t) = t + it^2 \qquad z'(t) = 1 + 2it$$

$$\int_\Gamma \bar{z}\, dz = \int_0^1 (t - it^2)(1 + 2it)\, dt = \int_0^1 t\, dt + \int_0^1 -it^2\, dt + \int_0^1 2it^2\, dt + \int_0^1 2t^3\, dt$$

$$= \frac{1}{2} - \frac{1}{3}i + \frac{2i}{3} + \frac{1}{2} = 1 + \frac{i}{3}.$$

3. [15 pts] Integrate the following functions over the set $|z| = 1$, traversed in the positive direction:

   i. $\dfrac{e^{2z}}{4z - i\pi}$ (10 pts)

   ii. $\dfrac{13z + 2}{(z^2 - 4)^2}$ (5 pts)

i. $$\int_{|z|=1} \frac{e^{2z}}{4z - i\pi}\, dz = \frac{1}{4}\int_{|z|=1} \frac{e^{2z}}{z - i\pi/4}\, dz = \frac{1}{4} \cdot 2\pi i \cdot e^{2 \cdot i\pi/4} = -\frac{\pi}{2}$$

Cauchy int. formula

ii. $$\int_{|z|=1} \frac{13z+2}{(z^2-4)^2}\, dz = 0$$ by Cauchy's thm.

4. [10 pts] Compute the integral

$$\int_{\sigma_{2019}} \frac{e^{iz}}{z^2+1}\, dz,$$

where, σ_{2019} is the counterclockwise semicircle formed by the segment $[-2019, 2019]$ on the real axis, followed by the circular arc of radius 2019 in the upper half plane from 2019 to -2019.

$$\int_{\sigma_{2019}} \frac{e^{iz}}{z^2+1}\, dz = \int_{|z-i|=1} \frac{e^{iz}}{z^2+1}\, dz$$

$$= \int_{|z-i|=1} \frac{e^{iz}/(z+i)}{z-i}\, dz = 2\pi i \cdot \frac{e^{ii}}{i+i}$$

$$= \frac{\cancel{2}\pi \cancel{i}\, e^{-1}}{\cancel{2}\cancel{i}} = \frac{\pi}{e}$$

5. [10 pts] Let f be an entire function that is *bounded* by 2019. That is, f is analytic in $\mathbb{C}$, and $|f(z)| \leq 2019$ for all $z \in \mathbb{C}$. Prove that f must be a constant function.

Hint: For an arbitrary point $a \in \mathbb{C}$, integrate a carefully selected function over the set $|z-a| = R$ to obtain the estimate $|f'(a)| \leq \frac{2019}{R}$.

By Cauchy's estimate with $n = 1$,

$$|f'(a)| \leq \frac{1}{R} \sup_{|\zeta - a| = R} |f(\zeta)| \leq \frac{2019}{R}$$

so sending $R \to \infty$ shows that $f'(a) = 0$.

Since a was arbitrary, $f' \equiv 0$ so that f is constant.

132 disc. 9

Local power series representation. Suppose f is analytic in Ω, $z_0 \in \Omega$, $z \in B_r(z_0) \subseteq \Omega$:

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

$$= \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \, d\zeta$$

$\left|\frac{z-z_0}{\zeta - z_0}\right| < 1$

$$= \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n d\zeta$$

$$\overset{\text{unif.}}{=} \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta\right)(z - z_0)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{n!}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta\right) \frac{(z - z_0)^n}{n!}$$

so

- f can be expressed as a power series in $B_r(z_0)$
- f is infinitely differentiable
- The derivatives at z_0 are given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta .$$

Now if $M = \max_{|\zeta - z_0| = r} |f(\zeta)|$

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \int_{|\zeta - z_0| = r} \frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}} |d\zeta| \leq \frac{n! M}{2\pi} \cdot \frac{2\pi r}{r^{n+1}} = \frac{n! M}{r^n}$$

$$|f^{(n)}(z_0)| \leq \frac{n! M}{r^n} \qquad \text{(Cauchy's estimates)}$$

Cor. (Liouville's thm) Every bounded entire function is constant.

If f is bounded & entire, $\sup_{|\zeta - z_0| = r} |f(\zeta)| \leq M$ for some M, all $r > 0$, all $z_0 \in \mathbb{C}$.

By Cauchy's est. ($n = 1$), $|f'(z_0)| \leq \frac{M}{r}$ $(r > 0)$.

Sending $r \to \infty \Rightarrow f'(z_0) = 0$. Since z_0 was arbitrary, $f' \equiv 0$.

Prob. Suppose f is entire and $|f(z)| \le C|z|^n$ for $|z|$ large. Then f is a polynomial of deg. $\le n$.

Pf. By Cauchy's ests.,

$$|f^{(n+k)}(z_0)| \le \frac{(n+k)!}{r^{n+k}} \sup_{|\zeta - z_0| = r} |f(\zeta)|$$

$$\le \sup_{''} \frac{(n+k)!}{r^{n+k}} C|\zeta|^n \le C_2 \frac{(n+k)!}{r^{n+k}} r^n$$

$$= C_2 (n+k)! \, r^{-k}$$

so for $k > 0$, sending $r \to \infty$ shows that

$$f^{(n+k)}(z_0) = 0. \quad (*)$$

By the power series repr.,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

So by $(*)$,

$$f(z) = f(0) + f'(0) z + \frac{f''(0)}{2!} z^2 + \cdots + \frac{f^{(n)}(0)}{n!} z^n,$$

so f is a polynomial of degree $\le n$.

Problem 11. Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is a holomorphic function such that the function $z \mapsto g(z) = f(z)f(1/z)$ is bounded on $\mathbb{C} \setminus \{0\}$.

(a) Show that if $f(0) \neq 0$, then f is constant.

(b) Show that if $f(0) = 0$, then there exist $n \in \mathbb{N}$ and $a \in \mathbb{C}$ such that $f(z) = az^n$ for all $z \in \mathbb{C}$.

(a) If $f(0) \neq 0$, the bound gives

$$|f(1/z)| = \left| \frac{g(z)}{f(z)} \right| \le \frac{\|g\|_\infty}{|f(z)|} \quad \text{for } |z| \text{ small},$$

hence f is bounded for $|z|$ large.

By Liouville's thm., f is constant.

(b) Write $f(z) = z^n h(z)$, where $h(0) \neq 0$. Then

$$g(z) = z^n h(z) z^{-n} h(1/z) = h(z) h(1/z),$$

so part (a) shows that h is constant, say $\equiv a$.

Thus $$f(z) = z^n h(z) = a z^n.$$

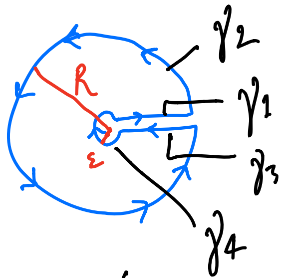
132 disc. 10 Example residue theorem application

Suppose $a, b > 0$. We will evaluate

$$\int_0^\infty \frac{\log x}{(x+a)^2+b^2}\,dx.$$

$=\!=$

Define a keyhole contour Γ:



We consider $$\int_\Gamma \frac{\log^2 z}{(z+a)^2+b^2}\,dz = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}\right)\frac{\log^2 z\,dz}{(z+a)^2+b^2}$$

where we define log with a branch cut on $\mathbb{R}^+$, so that $0 < \arg z < 2\pi$.

We have
$$\left|\int_{\gamma_2} \cdots\right| = O\left(\frac{\log^2 R}{R^2}\right) \to 0 \text{ as } R \to \infty,$$

& sim. for γ_4 as $\varepsilon \to 0$.

Moreover,
$$\int_{\gamma_1} \cdots = \int_\varepsilon^R \frac{\log^2 x}{(x+a)^2+b^2}\,dx, \qquad \int_{\gamma_3} \cdots = \int_R^\varepsilon \frac{(\log x + 2\pi i)^2}{(x+a)^2+b^2}\,dx,$$

so $$\int_{\gamma_1} + \int_{\gamma_3} \cdots = \int_\varepsilon^R \frac{\log^2 x - (\log^2 x + 4\pi i \log x - 4\pi^2)}{(x+a)^2+b^2}\,dx.$$

On the other hand, $\dfrac{\log^2 z}{(z+a)^2+b^2}$ has simple poles at $\pm ib - a$,

so by the Residue Theorem,

$$\int_\Gamma \frac{\log^2 z}{(z+a)^2+b^2}\,dz = 2\pi i\left(\operatorname*{Res}_{z=ib-a}\frac{\log^2 z}{(z+a)^2+b^2} + \operatorname*{Res}_{z=-ib-a}\frac{\log^2 z}{(z+a)^2+b^2}\right)$$

$$= 2\pi i \lim_{z\to ib-a}\frac{\log^2 z}{z-(-ib-a)} + 2\pi i\lim_{z\to -ib-a}\frac{\log^2 z}{z-(ib-a)}$$

$$= \cancel{2\pi i}\,\frac{\log^2(ib-a)}{\cancel{2i}b} + \cancel{2\pi i}\,\frac{\log^2(-ib-a)}{-\cancel{2i}b}$$

$$= \frac{\pi}{b}\left(\log^2(ib-a) - \log^2(-ib-a)\right),$$

so that

$$-4\pi\int_0^\infty \frac{\log x}{(x+a)^2+b^2}\,dx = \frac{\pi}{b}\,\mathrm{Im}\left(\log^2(ib-a) - \log^2(-ib-a)\right),$$

(usual real log)

where $\log^2 z = (\log|z| + i\arg z)^2$

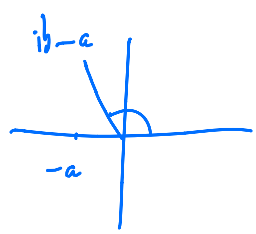
$$= \log^2|z| + 2i\arg z\log|z| - \arg^2 z,$$

hence

$$(*)\quad \log^2(ib-a) - \log^2(-ib-a) = \log^2|ib-a| + 2i\arg(ib-a)\log|ib-a|$$
$$-\arg^2(ib-a)$$
$$-\log^2|-ib-a| - 2i\arg(-ib-a)\log|-ib-a|$$
$$+\arg^2(-ib-a),$$

where $\arg(ib-a) = \pi - \arg(a+ib) = \pi - \arctan\left(\frac{b}{a}\right)$

& $\arg(-ib-a) = \pi + \arctan\left(\frac{b}{a}\right)$.



Thus,

$$\mathrm{Im}((*)) = 2\arg(ib-a)\log\sqrt{a^2+b^2} - 2\arg(-ib-a)\log\sqrt{a^2+b^2}$$

$$= \left[\left(\pi - \arctan\frac{b}{a}\right) - \left(\pi + \arctan\frac{b}{a}\right)\right]\cdot\log(a^2+b^2)$$

$$= -2\arctan\left(\frac{b}{a}\right) \cdot \log(a^2+b^2),$$

hence

$$\int_0^\infty \frac{\log x}{(x+a)^2+b^2}\,dx = + \frac{1}{4\cancel{\pi}} \cdot \frac{\cancel{\pi}}{b} \cdot +2\arctan\left(\frac{b}{a}\right)\log(a^2+b^2)$$

$$= \frac{1}{2b}\arctan\left(\frac{b}{a}\right) \cdot \log(a^2+b^2).$$

For example, if $a = b = 1$,

$$\int_0^\infty \frac{\log x}{x^2+2x+2}\,dx = \frac{1}{2}\arctan 1 \log 2$$

$$= \frac{\pi}{8}\log 2.$$