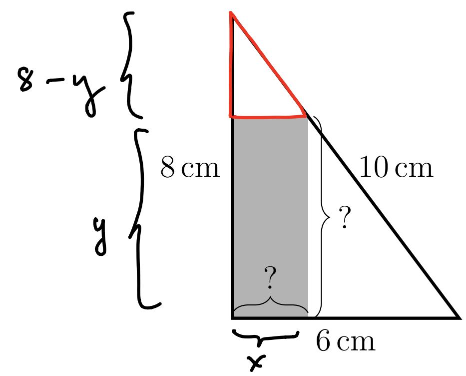
Math 31AL Worksheet
Tuesday, Nov 19 (Week 8)

1. In this problem, you will find the maximum area of a rectangle inscribed in a right triangle whose sides have lengths 6 cm, 8 cm, and 10 cm. Assume that two of the edges of the rectangle lie along the legs of the triangle.

   (a) Identify some variables:



   (b) In terms of your variables, what quantity are you trying to *maximize*?

$$A = xy$$

   (c) Come up with an equation relating your variables. *(Hint: Similar triangles.)*

Comparing the red triangle with the whole triangle: $\frac{8-y}{x} = \frac{8}{6}$

   (d) Solve your equation from part (c) and substitute it into the expression from part (b) to get a function that's in terms of just one variable. This is the function you want to maximize.

$$\frac{3}{4}(8-y) = x$$
$$\Rightarrow \quad A = \frac{3}{4}(8-y)y$$

   (e) Maximize the function you found. (Be sure to verify that your solution is indeed a maximum by using the number line test to see where the function is increasing and decreasing.)

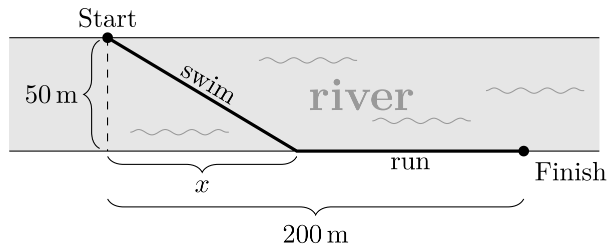
$$0 = A'(y) = \frac{3}{4}\cdot 8 - \frac{3}{4}\cdot 2y$$
$$\Rightarrow \quad 0 = 6 - \frac{3}{2}y$$
$$\Rightarrow \quad y = 4 \text{ cm}$$

Hence $x = \frac{3}{4}(8-y) = 3$ cm & the area is 12 cm^2.

(Note that this must be a max. b/c the function is a downward parabola, or see directly that it's concave down, or check the sign of A' on both sides of $y = 4$.)

2. You are standing on one side of a river that is 50 m wide. On the other side of the river and 200 m downstream, an adorable puppy is about to fall in the water and drown! In this problem you will find the route that gets you to the puppy in the *least* time possible, so you can save him from drowning. Your route will involve swimming diagonally across the river at a constant speed of 1.5 $\frac{\text{m}}{\text{s}}$, then running along the shore at a constant speed of 4 $\frac{\text{m}}{\text{s}}$. (Ignore the current of the river.)

   (a) There are several ways to choose variables for this problem. Let's do everything in terms of the variable x as shown in the picture. Find the distance you will run, in terms of x. Then, using distance = rate · time, find the *time* it will take you to run this distance, also in terms of x.



$$\text{dist} = 200 - x$$
$$4 \cdot t_1 = 200 - x$$
$$\Rightarrow t_1 = 50 - \frac{x}{4}$$

   (b) What is the distance you will swim, in terms of x? Use distance = rate · time again to find, in terms of x, the *time* it will take you to swim this distance.

$$\text{dist} = \sqrt{50^2 + x^2}$$
$$1.5 t_2 = \text{dist} \quad \Rightarrow \quad t_2 = \frac{2}{3}\sqrt{50^2 + x^2}$$

   (c) Using your results from parts (a) and (b), what is the total time it will take you to reach the puppy, in terms of x? This is the function you want to minimize.

$$T = t_1 + t_2 = 50 - \frac{x}{4} + \frac{2}{3}\sqrt{50^2 + x^2}$$

   (d) Minimize the function you found in part (c). (Be sure to verify that your solution is indeed a maximum by using the number line test to see where the function is increasing and decreasing.)

$$0 = T'(x) = -\frac{1}{4} + \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}} \cdot (50^2 + x^2)^{-1/2} \cdot 2x$$
$$\Rightarrow \frac{1}{4}(50^2 + x^2)^{1/2} = \frac{2}{3}x$$
square
$$\Rightarrow \frac{1}{4^2}(50^2 + x^2) = \left(\frac{2}{3}\right)^2 x^2$$
$$\Rightarrow 50^2 + x^2 = \left(\frac{8}{3}\right)^2 x^2$$
$$\Rightarrow 50^2 = \left(\left(\frac{8}{3}\right)^2 - 1\right) x^2$$
$$\Rightarrow x = \frac{50}{\sqrt{(8/3)^2 - 1}} \qquad (\text{we know } x > 0)$$
$$\approx 20.226 \text{ m}$$

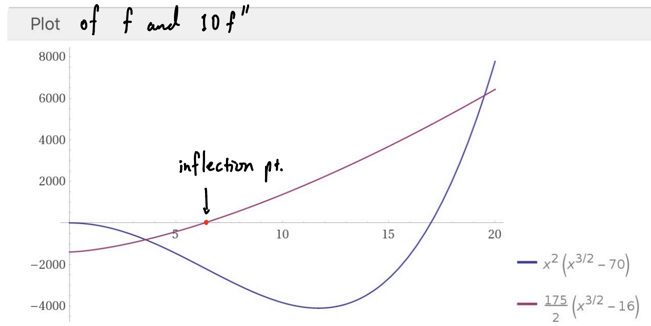
3. Consider the function $f(x) = x^{7/2} - 70x^2$

(a) Find the derivative f' and the second derivative f''.

$$f'(x) = \frac{7}{2} x^{5/2} - 140x$$

$$f''(x) = \frac{7}{2} \cdot \frac{5}{2} x^{3/2} - 140$$

$$= \frac{35}{4} x^{3/2} - 140$$



(b) Find the points where $f''(x)$ is undefined or 0.

f'' is defined whenever $x \geq 0$.

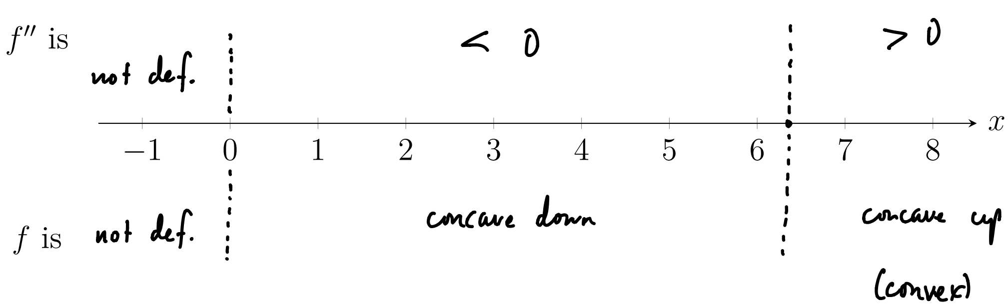
$$f'' = 0 \Rightarrow \frac{35}{4} x^{3/2} = 140 \Rightarrow x^{3/2} = 140 \cdot 4/35$$

$$\Rightarrow x \approx 6.3496$$

(c) Do the number line test on f'' to determine the intervals on which f is concave up and concave down.

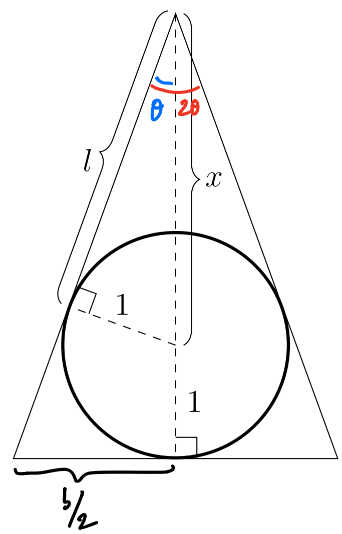
$$f''(7) = (35/4)\, 7^{3/2} - 140 \approx 22.05 > 0$$

$$f''(1) = 35/4 - 140 < 0$$



4. In this problem, you will find the dimensions of the isosceles triangle with the smallest area that can contain a circle of radius 1 inside it. (Note: This problem is hard. It's probably the most difficult problem from this week's homework.)

   (a) There are many different ways to set up this problem, mainly based on how you choose your variables. Probably the easiest way is as shown in the diagram below. First, find an equation that relates x and l. (This will be your *constraint* equation.) Solve this for l in terms of x.



$$x^2 = 1^2 + l^2$$

$$\Rightarrow \quad l = \sqrt{x^2 - 1}$$

(note: $l > 0$, $x \geq 1$)

   (b) Recall that area of a triangle is $\frac{1}{2} \cdot$ base $\cdot$ height. Write down the *height* of the triangle in terms of x. Then find the length of *half the base* of the triangle, in terms of x. *(Hint: Similar triangles again.)*

$$h = 1 + x$$

$$\frac{b/2}{1+x} = \frac{1}{l}$$

   (c) Use your results from parts (a) and (b) to find the area of the triangle in terms of x. This is the function you want to minimize.

$$A = h \cdot \left(\frac{b}{2}\right) = (1+x)\,\frac{1}{l}(1+x) = \frac{(1+x)^2}{\sqrt{x^2-1}}$$

   (d) Minimize the function you found in part (c). Based on the ~~angle~~ length x that gives the minimum area, what kind of triangle is this/what are its dimensions?

$$0 = A' = \frac{d}{dx}\left((1+x)^2 (x^2-1)^{-1/2}\right)$$

$$= 2(1+x)(x^2-1)^{-1/2} - \frac{1}{\cancel{2}}(x^2-1)^{-3/2}(\cancel{2}x)(1+x)^2$$

$$\Rightarrow \quad 0 = 2 - x(x^2-1)^{-1}(1+x)$$

$$\Rightarrow \quad \frac{x(1+x)}{x^2-1} = 2$$

$$\Rightarrow \quad x(1+x) = 2(x^2-1)$$

$$\Rightarrow \quad x + x^2 = 2x^2 - 2$$

$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad x = 2 \quad (\text{know } x > 0)$$

Note that

$$\sin(\theta) = \frac{1}{x} = \frac{1}{2}$$

$$\Rightarrow \quad 2\theta = 60^\circ$$

so the triangle is <u>equilateral</u>.

Note: we can do the problem in terms of θ:

$$\sin\theta = \frac{1}{x} \Rightarrow x = \csc\theta \Rightarrow h = 1 + \csc\theta$$

$$\frac{b/2}{1+x} = \tan\theta \Rightarrow \frac{b}{2} = (1+\csc\theta)\tan\theta$$

So the area is: $A = h\left(\frac{b}{2}\right) = (1+\csc\theta)^2 \tan\theta$

Hence at a loc. extremum:

$$0 = A'(\theta) = -2(1+\csc\theta)(\sin\theta)^{-2}\cos\theta\tan\theta + (1+\csc\theta)^2\sec^2\theta$$

$$\Rightarrow 2\frac{1}{\sin^{\cancel{2}}\theta}\cancel{\cos\theta}\frac{\cancel{\sin\theta}}{\cancel{\cos\theta}} = \left(1+\frac{1}{\sin\theta}\right)\frac{1}{\cos^2\theta}$$

$$\Rightarrow 2 = \frac{\sin\theta}{\cos^2\theta}\left(1+\frac{1}{\sin\theta}\right) = \frac{\sin\theta}{\cos^2\theta} + \frac{1}{\cos^2\theta}$$

$$\Rightarrow 2\cos^2\theta = \sin\theta + 1$$

$$\Rightarrow 2(1-\sin^2\theta) = \sin\theta + 1$$

$$\Rightarrow 2 - 2\sin^2\theta = \sin\theta + 1$$

$$\Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$$

$$\Rightarrow \sin\theta = \frac{-1 \pm \sqrt{1-4\cdot 2(-1)}}{4} = \frac{1}{2}, \cancel{-1}$$

(we know $0 < \theta < \pi/2$)

$$\Rightarrow \theta = 30^\circ.$$

Hence the triangle is equilateral.