Math 31AL Worksheet
Tuesday, Nov 5 (Week 6)

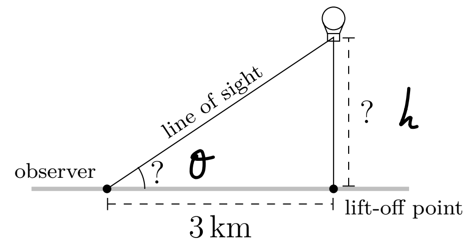
Guidelines for these problems:

1. Identify **variables**. Note which ones are functions of time (i.e. actual variables, *varying* with time, as opposed to just constants). It may help to draw a picture.
2. What is **given** in the problem? (Try to restate in terms of the variables, from step 1.)
3. What do you need to **find**? (Try to restate in terms of the variables.)
4. Come up with an **equation** that relates the variables to each other.
5. **Solve**: Differentiate both sides of the equation *with respect to t*, applying the chain rule, product rule, etc, wherever necessary. Plug in any values given in the problem.

Notes:

- One (or more) of the givens in the problem will be a rate of change of some variable.
- The thing you are supposed to find will usually also be a rate of change, of a different variable.
- Remember, if you're given a specific value of *one of the variables that's changing with time*, **don't plug that value in until *after* taking the derivative** in step 5.

1. A hot air balloon rising vertically is tracked by an observer located 3 km from the lift-off point. At a certain moment, the angle between the observer's line of sight and the horizontal is $\frac{\pi}{3}$, and it is changing at a rate of $0.1\,\frac{\text{rad}}{\text{min}}$. How fast is the hot air balloon rising at this moment?



(a) Identify two significant variables. *(Hint: See picture.)*

h, θ

(b) What, in terms of the variables you listed in part (a), is given in the problem?

$$\frac{d\theta}{dt} = 0.1 \ \text{rad}/\text{min}, \qquad \theta = \pi/3 \ \text{rad}$$

(c) What, in terms of the variables you listed in part (a), are you trying to find?

$$\frac{dh}{dt}$$

(d) Write down an equation relating the two variables. *(Hint: See the picture again.)*

$$\tan\theta = \frac{h}{3}$$

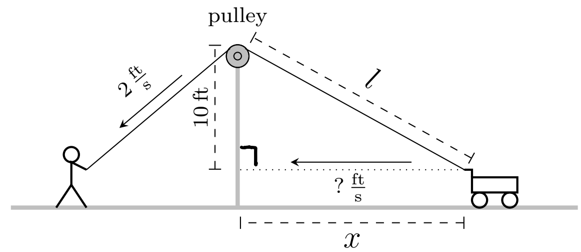
(e) Differentiate both sides of your equation from part (d), *with respect to t*. (Remember that you're thinking of the variables as *functions of t*. Then plug in numbers and solve.

$$(\sec^2\theta(t))\,\theta'(t) = \frac{1}{3}h'(t)$$

$$\Rightarrow h' = 3 \cdot 0.1 \cdot \sec^2\frac{\pi}{3} \ \text{km}/\text{min}$$

$$= 1.2 \ \text{km}/\text{min}$$

2. Lucciano uses a rope, running through a pulley at the top of a pole, to pull a wagon towards him. (See the diagram.) He pulls the rope through the pulley at a rate of $2\,\frac{\text{ft}}{\text{s}}$. Find the speed of the wagon. Your answer will be in terms of x.



$\longrightarrow$ $+x$

(a) In this case, the two variables you need are labeled in the diagram:

$$x = \text{the position of the wagon, i.e. its distance from the pole}$$
$$l = \text{the length of rope between the wagon and the pulley}$$

(b) What, in terms of these two variables, is given in the problem? *(Hint: If Lucciano is pulling on his end of the rope at $2\,\frac{\text{ft}}{\text{s}}$, what else is changing at the same rate?)*

$$\frac{dl}{dt} = -2 \ \text{ft}/\text{s}$$

(c) What, in terms of the variables in part (a), are you trying to find?

$$\left|\frac{dx}{dt}\right|$$

(d) Write down an equation relating the two variables. *(Hint: See the picture again.)*

$$x^2 + 10^2 = l^2$$

(e) Differentiate both sides of your equation from part (d), *with respect to t*. (Remember that you're thinking of the variables as *functions of t*. Then plug in numbers and solve.

$$2xx' = 2ll'$$

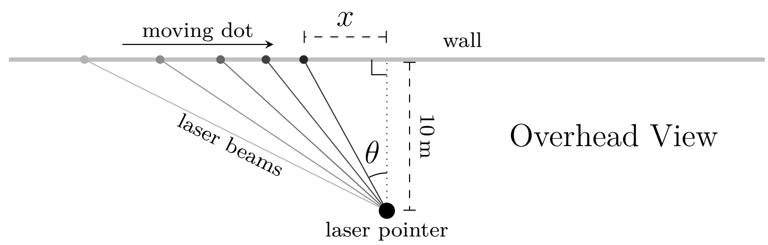
$$\Rightarrow \quad x' = \frac{l}{x} \cdot l' = -\frac{2l}{x} = -\frac{2}{x}\sqrt{100 + x^2} \ \text{ft}/\text{s}$$

So the speed is $\quad s = |x'| = \frac{2}{x}\sqrt{100 + x^2} \ \text{ft}/\text{s}$

Side note: $\quad \lim_{x \to \infty} \left(-2\sqrt{\frac{100}{x^2} + 1}\right) = -2$

So "speed at $t = -\infty$" $= 2$ ft/s for an infinitely long rope
(Which makes sense, since the angle is negligible if x is large)
On the other hand, $|x'| \to \infty$ as $x \to 0^+$

3. A laser pointer is placed on a platform that rotates at a rate of 30 revolutions per minute. The beam hits a wall 10 m away, producing a dot of light that moves horizontally along the wall. Let θ be the angle shown in the figure. How fast is this dot moving when $\theta = \frac{\pi}{4}$?



(a) In this case, the two variables you need are labeled in the diagram:

$$x = \text{the position of the dot on the wall}$$
$$\theta = \text{the angle of the laser pointer as it rotates}$$

(b) What quantity does the 30 revolutions per minute that's given in the problem represent? *(Hint: You'll have to convert revolutions per minute into radians per minute, or radians per second.)*

$$\frac{d\theta}{dt} = \frac{-30 \text{ rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{\text{rev}} = 60\pi \ \frac{\text{rad}}{\text{min}}$$

(c) What, in terms of the variables in part (a), are you trying to find?

$$\left| \frac{dx}{dt} \right|$$

(d) Write down an equation relating the two variables. *(Hint: See the picture again.)*

$$\tan\theta = \frac{x}{10}$$

(e) Differentiate both sides of your equation from part (d), *with respect to t.* (Remember that you're thinking of the variables as *functions of t.* Then plug in numbers and solve.

$$(\sec^2 \theta(t))\, \theta'(t) = \frac{1}{10} x'(t)$$

$$\Rightarrow x' = 10 \cdot (-60\pi) \cdot \sec^2 \pi/4 \quad m/\text{min}$$

$$|x'| \approx 3.77 \ \text{km}/\text{min}$$