Solns. to

Math 31AL Worksheet
Tuesday, Oct 15 (Week 3)

1. Finish the following two standard definitions of $f'(a)$, the derivative of a function f at $x = a$, using limits:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{2}$$

(You can see these are the same by taking $h = x - a$)

2. Let $f(x) = 5x^2 + 3x + 4$, and let $a = -1$.

   (a) What is $f(a)$, that is, $f(-1)$?

   $= 5 - 3 + 4 = 6$

   (b) Write down $f(a+h)$, that is, $f(-1+h)$.

   $$f(-1+h) = 5(-1+h)^2 + 3(-1+h) + 4$$

   (c) Use equation (2) above (the $h \to 0$ version of the definition of the derivative), with help from parts (a) and (b), to compute $f'(-1)$.
   (Note: You may use the Power Rule to check that your answer is correct, but you should compute this derivative from the limit definition!)

$$\frac{f(-1+h) - f(-1)}{h} = \frac{5(1 - 2h + h^2) - 3 + 3h + 4 - 6}{h} = \frac{\overbrace{\cancel{5 - 3 + 4 - 6}}^{0} - 10h + h^2 + 3h}{h}$$

$$= -7 + h \longrightarrow -7 \text{ as } h \to 0.$$

(Using the power rule, $f'(x) = 10x + 3 \Rightarrow f'(-1) = -7$)

3. Let $f(x) = \frac{1}{x}$, and let $a = 2$.

(a) What is $f(a)$, that is, $f(2)$? $\frac{1}{2}$

(b) Write down $\dfrac{f(x) - f(a)}{x - a}$, that is, $\dfrac{f(x) - f(2)}{x - 2}$.

$$\frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$$

(c) Now use equation (1) above (the $x \to a$ version of the definition of the derivative), with help from part (b), to compute $f'(2)$.

(Note: You may use the Power Rule to check that your answer is correct, but you should compute this derivative from the limit definition!)

Note that (for $x \neq 2$)
$$\frac{\frac{1}{x} - \frac{1}{2}}{x - 2} = \frac{\frac{2}{2x} - \frac{x}{2x}}{x - 2} = \frac{\frac{2-x}{2x}}{x - 2} = \frac{2 - x}{2x(x-2)}$$

$$= \frac{-(x-2)}{2x(x-2)} = -\frac{1}{2x}$$

so $$f'(2) = \lim_{x \to 2} \left(-\frac{1}{2x}\right) = -\frac{1}{4}.$$

$$\left(f(x) = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f'(2) = -\frac{1}{4} \right)$$

4. Compute the following limits:

(a)
$$\lim_{h \to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{\sqrt{4}}}{h} = \lim_{h \to 0} \frac{\left(\frac{1}{\sqrt{4+h}} - \frac{1}{\sqrt{4}}\right)}{h} \cdot \frac{\left(\frac{1}{\sqrt{4+h}} + \frac{1}{\sqrt{4}}\right)}{\left(\frac{1}{\sqrt{4+h}} + \frac{1}{\sqrt{4}}\right)} = \lim_{h \to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h\left(\frac{1}{\sqrt{4+h}} + \frac{1}{\sqrt{4}}\right)}$$

we want to eliminate this.

$$= \lim_{h \to 0} \frac{\frac{4-(4+h)}{4(4+h)}}{h\left(\frac{1}{\sqrt{4+h}} + \frac{1}{\sqrt{4}}\right)} = \lim_{h \to 0} \frac{\frac{-\cancel{h}}{4(4+h)}}{\cancel{h}\left(\frac{1}{\sqrt{4+h}} + \frac{1}{\sqrt{4}}\right)} = \frac{\frac{-1}{4\cdot(4+0)}}{\frac{1}{\sqrt{4+0}} + \frac{1}{\sqrt{4}}} = -\frac{1}{16}.$$

Note that $(h-1)^4 = h^4 - 4h^3 + 6h^2 - 4h + 1$

(b)
$$\lim_{h \to 0} \frac{\frac{1}{(-1+h)^2} - \frac{1}{(-1)^2}}{h} = \lim_{h \to 0} \frac{\left(\frac{1}{(-1+h)^2} - 1\right)\left(\frac{1}{(-1+h)^2} + 1\right)}{h\left(\frac{1}{(-1+h)^2} + 1\right)} = \lim_{h \to 0} \frac{1 - (-1+h)^4}{h\left(\frac{1}{(-1+h)^2} + 1\right)(-1+h)^4}$$

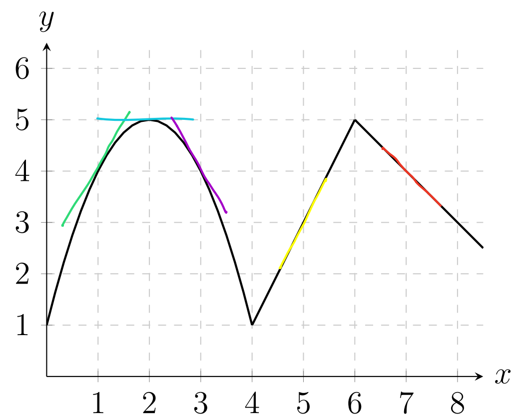
$$= \lim_{h \to 0} \frac{-h^{\cancel{4}\,3} + 4h^{\cancel{3}\,2} - 6h^{\cancel{2}} + 4\cancel{h}}{\cancel{h}\left(\frac{1}{(-1+h)^2} + 1\right)(-1+h)^4} = \frac{4}{\left(\frac{1}{(-1+0)^2} + 1\right)(-1+0)^4} = 2$$

(c) Each of the previous two limits represents the instantaneous rate of change of a function $f(x)$ at some point $x = a$. For each one, what was the function $f(x)$, and at what point were you finding the instantaneous rate of change?

For (a), $f(x) = \frac{1}{\sqrt{x}}$ at $x = 4$

For (b), $f(x) = \frac{1}{x^2}$ at $x = -1$

5. Consider the following graph of a function f.



(a) On the graph above, sketch the tangent lines at $x = 1$, $x = 2$, $x = 3$, $x = 5$, and $x = 7$.

(b) What is $f'(5)$? What is $f'(7)$? What is $f'(2)$?

$= 2$ $\quad$ $= -1$ $\quad$ $= 0$

(c) For what range(s) of x values on the graph is $f'(x)$ negative?

$(2, 4)$ and $(6, 8.5)$ (about)

(d) Explain why $f'(6)$ does not exist, in two different ways: (1) by saying something about the tangent line, and (2) using the limit definition of the derivative. *(Hint: Think about the one-sided limits, from the left and from the right.)*

(1) For the derivative to exist, the slope of the tangent line at a for $a \to 6$ from the left and the slope of the tangent line at a for $a \to 6$ from the right must approach the same value.

But the former is 2 and the latter is -1 (for a near 6).

(2) The derivative is a limit, so the left- and right-hand limits must agree. But we see that the left limit is 2 and the right limit is -1.

(e) At what other value of x does $f'(x)$ not exist?

$x = 4$